LMI-Based Fuzzy Optimal Variance Control of Airfoil Model Subject to Input Constraints

Sean S.M. Swei
Research Scientist
Intelligent Systems Division
NASA Ames Research Center
Moffett Field, California 94035
Email: sean.s.swei@nasa.gov

Mohammad A. Ayoubi
Associate Professor
Mechanical Engineering Department
Santa Clara University
Santa Clara, California 95053
Email: maayoubi@scu.edu

***Abstract*—This paper presents a study of fuzzy optimal variance control problem for dynamical systems subject to actuator amplitude and rate constraints. Using Takagi-Sugeno fuzzy modeling and dynamic Parallel Distributed Compensation technique, the stability and the constraints can be cast as a multi-objective optimization problem in the form of Linear Matrix Inequalities. By utilizing the formulations and solutions for the input and output variance constraint problems, we develop a fuzzy full-state feedback controller. The stability and performance of the proposed controller is demonstrated through its application to the airfoil flutter suppression.**

I. INTRODUCTION

Control of dynamical systems with bounded control input has been studied for decades [1]–[5]. In practical applications, it has been shown that the actuator rate limit is the primary source of instability and performance degradation, especially for aircraft control systems, e.g. see [2] and [4]. Previous works on bounded input and input rate; see [1]–[3] and the references therein, offer a good framework for control implementation. A simple "software limiter" idea was proposed in Ref. [2]. There, by introducing the derivative and limited integration blocks, a commanded input signal can be re-generated which satisfies both input and input rate constraints. In Ref. [3] a bounded input and input rate problem description was given that, to an extent, prescribes how the actuator should behavior. An extended high performance bounded control algorithm was proposed in Ref. [1], which was used to suppress the vibrational motion of the aircraft wing subject to bounded actuation on position/rate. All these results were obtained by considering only one given dynamical system description. For a control system that contains scenario dependent (varying) parameters, multiple system descriptions can be generated based on each parameter value. In this case, one common approach that is used to address the gain scheduling and stability/performance issues is the Linear Parameter Varying (LPV) method [6], [7]. In the past three decades, the theory of fuzzy control systems has been developed tremendously. In particular, the stability and performance of a fuzzy control problem can be cast as solving a feasibility problem in the form of Linear Matrix Inequalities (LMIs) [8]–[11]. Therefore, the numerically efficient LMI-based convex optimization tools and algorithms can readily be utilized for control system analysis and synthesis.

In this paper, we consider the problem of fuzzy optimal variance control for a parameter varying (nonlinear) "smart" airfoil model [12], as shown in Fig. 1 where the actuator amplitude and rate are constrained. The Takagi-Sugeno (T-S) fuzzy model is used to approximate the parameter varying system via a set of convex combination of local linear models. We show that the bounded input and bounded input rate problem can be converted into a bounded output and bounded input problem by augmenting the control input as an additional state. We show that the T-S fuzzy optimal variance control problem is closely related to the input and output variance constrained problems [13], whose formulations and solutions are essential in attaining the main results of this paper. We present that the feasibility of the problem via a full-state feedback controller can be characterized using the LMIs for each local model.

II. PROBLEM STATEMENT

We consider a 2-D smart airfoil model as illustrated in Fig. 1. The airfoil is subjected to the free stream airflow at varying speed V, and is allowed to pitch (α) and plunge (h) about the supported axis. The rotational and translational spring constants are denoted by K_α and K_h, respectively, as shown in Fig. 1. A light ballast mass, m, which is placed along the airfoil chord, is electronically activated and served as controllable effector. The objective of this smart airfoil platform is to suppress the fluttering motion of the airfoil, subjected to varying free stream airspeed, through actuation of the ballast mass. However, the ballast motion is physically constrained by the length of its travel and the motor driving rate, and these constraints pose a practical control design challenge.

The equations of motion for this 2-D smart airfoil model can be described in the nondimensional form as follows [12],

$$\begin{bmatrix} 1+\beta & \bar{x}_\alpha \\ \bar{x}_\alpha & \bar{r}_\alpha^2 \end{bmatrix} \begin{bmatrix} \ddot{\bar{h}}(\tau) \\ \ddot{\alpha}(\tau) \end{bmatrix} + \begin{bmatrix} 2\bar{V}/\mu & 0 \\ -2\bar{V}\bar{e}/\mu & 0 \end{bmatrix} \begin{bmatrix} \dot{\bar{h}}(\tau) \\ \dot{\alpha}(\tau) \end{bmatrix} + \begin{bmatrix} \omega_h^2/\omega_\alpha^2 & 2\bar{V}^2/\mu \\ 0 & -2\bar{V}^2\bar{e}/\mu + \bar{r}_\alpha^2 \end{bmatrix} \begin{bmatrix} \bar{h}(\tau) \\ \alpha(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ \beta\bar{g} \end{bmatrix} \{\bar{y}(\tau) + d(\tau)\} \tag{1}$$

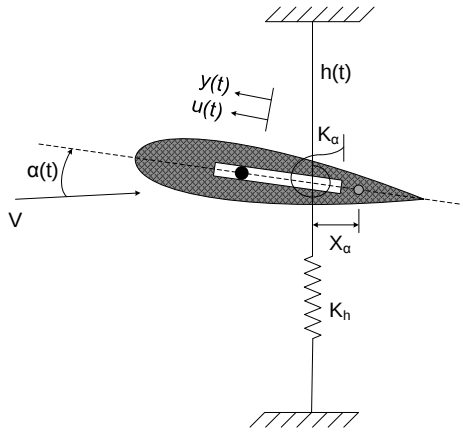

Fig. 1. The smart airfoil model

where $\tau = \omega_\alpha t$ denotes the nondimensional time, β the mass ratio between m and the mass of airfoil, b the airfoil semi-chord, $\bar{V} = V/\omega_\alpha b$ the nondimensional airspeed, $\bar{h} = h/b$ the nondimensional plunging displacement, $\bar{y} = y/b$ the nondimensional displacement, and $d(\tau)$ the nondimensional random disturbance input. Now, if we let $x = [\bar{h}\ \dot{\bar{h}}\ \alpha\ \dot{\alpha}]$, then (1) can be rewritten in the state-space form as follows,

$$\Sigma_{\bar{V}}: \quad \dot{x}(\tau) = A(\bar{V})x(\tau) + Bu(\tau) + Dd(\tau) \tag{2}$$

where

$$A(\bar{V}) = a\begin{bmatrix} 0 & 1/a & 0 & 0 \\ -(\omega_n/\omega_\alpha)^2\bar{r}_\alpha^2 & -\frac{2\bar{V}}{\mu}(\bar{r}_\alpha^2 + \bar{x}_\alpha\bar{e}) & -\frac{2\bar{V}^2}{\mu}(\bar{r}_\alpha^2 + \bar{x}_\alpha\bar{e}) + \bar{x}_\alpha\bar{r}_\alpha^2 & 0 \\ 0 & 0 & 0 & 1/a \\ (\omega_n/\omega_\alpha)^2\bar{x}_\alpha & \frac{2\bar{V}}{\mu}[\bar{x}_\alpha + (1+\beta)\bar{e}] & \frac{2\bar{V}^2}{\mu}[\bar{x}_\alpha + (1+\beta)\bar{e}] - (1+\beta)\bar{r}_\alpha^2 & 0 \end{bmatrix}, \tag{3}$$

$$B = \begin{bmatrix} 0 \\ -a\bar{x}_\alpha\beta\bar{g} \\ 0 \\ a(1+\beta)\beta\bar{g} \end{bmatrix}, \; D = B,$$

and $a = 1/[(1+\beta)\bar{r}_\alpha^2 - \bar{x}_\alpha^2]$. Since $A(\bar{V})$ is explicitly function of free stream airspeed, the dynamic response of $\Sigma_{\bar{V}}$ varies as function of $\bar{V}$. In addition, the control input $u(\tau)$ is bounded by the length of groove and $\dot{u}(\tau)$ is constrained by the allowable motor torque. More specifically, in this study, we consider that $u = [u_1, u_2, \cdots u_m]$ and $\dot{u} = [\dot{u}_1, \dot{u}_2, \cdots \dot{u}_m]$, and

$$|u_i| \le \sigma_i\,,\ |\dot{u}_i| \le \tau_i\,, \quad (i = 1, \ldots, m) \tag{4}$$

where (σ_i, τ_i) are known constants. Let the $\mathscr{L}_2$-norm of the random disturbance $d(\cdot)$ be given by

$$\|d\|_2^2 \triangleq \int_0^\infty d^T(\tau)d(\tau)d\tau\,,$$

and the weighted $\mathscr{L}_2$ disturbance set $\mathscr{W}$ be defined as

$$\mathscr{W} \triangleq \left\{ d : \mathbb{R} \to \mathbb{R}^d\,,\ \|W^{\frac{1}{2}}d\|_2^2 \le 1 \right\},$$

where W is a positive definite symmetric matrix. Let the $\mathscr{L}_\infty$-norm of $y(\cdot)$ be defined by

$$\|y\|_\infty^2 \triangleq \sup_{\tau \ge 0} y^T(\tau)y(\tau)\,.$$

Therefore, the input amplitude and rate constraints given in (4) are precisely the $\mathscr{L}_\infty$ bounds.

III. TAKAGI-SUGENO FUZZY MODEL

The power of T-S fuzzy modeling technique is that the nonlinear dynamical systems can be expressed as a convex combination of local linear systems through fuzzy IF-THEN rules which represent local input-output relations. It has been shown in [15] that smooth nonlinear systems can be adequately approximated by T-S fuzzy models. Before building a T-S fuzzy model, we first need to determine the premise variables and develop the corresponding model rules. Let $\theta(t)$ denote a vector of p known premise variables, i.e. $\theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_p(t)]$, then the number of model rules r is 2^p. For the purpose of this paper, we assume that θ is independent of system states and inputs. To this end, we may apply the above approximation to create a T-S fuzzy model for $\Sigma_{\bar{V}}$ and obtain a local linear model for each fuzzy rule as follows.

Model Rule i:
IF $\theta_1(t)$ is $\Xi_{i1}, \ldots,$ $\theta_p(t)$ is Ξ_{ip}, THEN:

$$\Sigma_i : \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) + D_i d(t) & (i = 1, 2, \ldots, r) \\ y(t) = C_i x(t) \end{cases} \tag{5}$$

where Ξ_{ij} is the fuzzy set, y the controlled output, and (A_i, B_i, C_i, D_i) are known constant matrices. We assume that the pair (A_i, B_i) is controllable. In addition, recall that each entry of u and $\dot{u}$ is bounded by (4).

The firing strength of each model rule can be determined using $T-norm$ product [11] as follows,

$$w_i(\theta) = \prod_{j=1}^{p} \Xi_{ij}(\theta) \tag{6}$$

and the fuzzy basis functions $h_i(\theta)$ are determined by

$$h_i(\theta) = \frac{w_i(\theta)}{\sum_{i=1}^r w_i(\theta)}\,;\ \sum_{i=1}^r h_i(\theta) = 1\,,\ h_i(\theta) \ge 0\,. \tag{7}$$

After combining all the rules of T-S models, the nonlinear system $\Sigma_{\bar{V}}$ can be approximated by convex combination of local models as

$$\Sigma_{TS} : \begin{cases} \dot{x} = \sum_{i=1}^r h_i(\theta)\,\{A_i x + B_i u + D_i d\} \\ y = \sum_{i=1}^r h_i(\theta)\,C_i x \end{cases} \tag{8}$$

Now, the *Full State Fuzzy Optimal Variance Control Problem* for $\Sigma_{\bar{V}}$ can be stated as follows:

> Given the input amplitude and rate constraints as in (4) and the initial condition constraint $\|x(0)\| \le \phi$, find a fuzzy full-state feedback controller so that it guarantees: i) internal stability for Σ_{TS}; ii) minimum output $\mathscr{L}_\infty$ performance (variance) cost.

In this paper, we introduce a convenient way of incorporating the effect of $\dot{u}$ by augmenting it into the system description, and this approach has been adopted throughout the paper. We introduce an augmented system description for ith model rule Σ_i as follows,

$$\Sigma_{ai} : \begin{cases} \dot{x}_a = A_{ai}x_a + B_a u_a + D_{ai}d \quad (i = 1,2,\ldots,r) \\ y_a = C_{ai}x_a \end{cases} \tag{9}$$

where

$$x_a = \begin{bmatrix} x \\ u \end{bmatrix}, \; u_a = \dot{u}, \; A_{ai} = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix}, \; B_a = \begin{bmatrix} 0 \\ I_m \end{bmatrix},$$
$$D_{ai} = \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \; y_a = \begin{bmatrix} y \\ u \end{bmatrix}, \; C_{ai} = \begin{bmatrix} C_i & 0 \\ 0 & I_m \end{bmatrix}.$$

Note that the new control input to the augmented system is the input rate $\dot{u}$. Furthermore, the input amplitude and rate constraints in (5) have now become the output and input constraints in the augmented system (9). Similarly, we can define the augmented T-S model for Σ_{TS} as follows,

$$\Sigma_{TSa} : \begin{cases} \dot{x}_a = \sum_{i=1}^{r} h_i(\theta)\left\{A_{ai}x_a + B_a u_a + D_{ai}d\right\} \\ y_a = \sum_{i=1}^{r} h_i(\theta) C_{ai}x_a \end{cases} \tag{10}$$

IV. SOME PRELIMINARY RESULTS

In this section, we present some preliminary results on LMI characterization for the input and output variance constrained problems. These results will play an instrumental role in obtaining the main theorem of this paper. We consider the following linear systems,

$$\begin{cases} \dot{x}(t) = Ax(t) + Dd(t) \\ y_i(t) = C_i x(t) \;\; (i = 1,2,\ldots,k_c) \end{cases} \tag{11}$$

where (C_i, A, D) are matrices of compatible dimensions and $d \in \mathscr{W}$. It is well understood that, if A is Hurwitz, the output variance for (11) is given by [16]

$$Y_i = C_i X C_i^T, \; (i = 1,2,\ldots,k_c), \tag{12}$$

where $X > 0$ is the controllability Gramian matrix satisfying the following Lyapunov equation

$$XA^T + AX + DWD^T = 0. \tag{13}$$

Therefore, for any $d \in \mathscr{W}$, the $\mathscr{L}_\infty$-norm of the output y_i is given by [17]

$$\|y_i\|_\infty^2 \le \sigma_{max}(Y_i), \tag{14}$$

where $\sigma_{max}(\cdot)$ denotes the maximum singular value. This indicates that the worst case output, $\|y_i\|_\infty$, is bounded by the squared root of the maximum singular value of output variance. Furthermore, given the desired output variance $\bar{Y}_i$, the output variance constrained problem for (11) is said to be solved when

$$Y_i \le \bar{Y}_i \;\; (i = 1,2,\ldots,k_c). \tag{15}$$

The following lemma completely characterizes the output variance constrained problem using the LMIs [13].

Lemma 1: Consider (11) and let $\bar{Y}_i > 0$, $i = 1,2,\ldots,k_c$, be the desired output variance. Then, the output variance constrained problem is solvable, if there exists $\bar{X} > 0$ satisfying

$$\text{(i)} \quad \begin{bmatrix} \bar{X}A^T + A\bar{X} & DW^{\frac{1}{2}} \\ W^{\frac{1}{2}}D^T & -I \end{bmatrix} < 0, \tag{16}$$
$$\text{(ii)} \quad C_i\bar{X}C_i^T < \bar{Y}_i, \;\; (i = 1,2,\ldots,k_c).$$

Next, we present the input variance constraint control problem [13]. To study this problem, we consider the following linear control systems,

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dd(t) \\ y_i(t) = C_i x(t), \; (i = 1,2,\ldots,k_c) \end{cases} \tag{17}$$

where the pair (A, B) is assumed controllable, and $d \in \mathscr{W}$. We consider the full state feedback control law of the form

$$u(t) = Gx(t), \tag{18}$$

such that $A_{cl} = (A + BG)$ is Hurwitz. Then, the input variance U for $u(t)$ is given by [13]

$$U = GXG^T, \tag{19}$$

where $X \ge 0$ solves the following Lyapunov equation

$$A_{cl}X + XA_{cl}^T + DWD^T = 0. \tag{20}$$

Note that the control input $u(t)$ can be partitioned into $u = [u_1, u_2, \cdots, u_m]^T$, and each u_i can be expressed as

$$u_i(t) = K_i x(t), \; G = \begin{bmatrix} K_1 & K_2 & \cdots & K_m \end{bmatrix}^T. \tag{21}$$

The $\mathscr{L}_\infty$-norm of input u_i is given by

$$\|u_i\|_\infty^2 \le K_i X K_i^T = \left\{GXG^T\right\}_{ii}, \;\; (i = 1,2,\ldots,m) \tag{22}$$

where $\{\}_{ii}$ denotes the ith diagonal entry. If we define $U_i = \left\{GXG^T\right\}_{ii}$ to be the ith input variance, then from (22) we obtain

$$\|u_i\|_\infty \le \sqrt{U_i}, \;\; (i = 1,2,\ldots,m) \tag{23}$$

that is, the maximum input is bounded by the squared root of input variance. Let $\bar{U}_i > 0$, $i = 1,2,\ldots,m$, be the given input variance for $u_i(t)$. Then, the input variance constraint control problem is solvable via a static full-state feedback controller of the form (21), if it minimizes the output variance cost

$$J = \text{trace}\left\{C_j X C_j^T\right\}, \;\; (j = 1,2,\ldots,k_c),$$

subject to: 1) $A_{cl} = (A + BG)$ is Hurwitz, and 2) $U_i = K_i X K_i^T \le \bar{U}_i$, $i = 1,2,\ldots,m$, where $X \ge 0$ solves (20). The next lemma characterizes the input variance constraint control problem using LMIs, whose proof follows readily from Lemma 1.

Lemma 2: Consider (17) and let $\bar{U}_i > 0$, $i = 1,2,\ldots,m$, be the given input variance. Then, the input variance constraint control problem via a static full-state feedback control is solvable, if there exist a matrix F and a positive definite symmetric matrix $\bar{X}$ that minimize

$$\min_{\bar{X},F} \text{trace}\left\{C_j\bar{X}C_j^T\right\}, \;\; (j = 1,2,\ldots,k_c), \tag{24}$$

subject to the following LMIs,

(i)

$$\begin{bmatrix} (\bar{X}A^T + A\bar{X} + F^T B^T + BF) & DW^{\frac{1}{2}} \\ W^{\frac{1}{2}} D^T & -I \end{bmatrix} < 0, \quad (25)$$

(ii)

$$\begin{bmatrix} \bar{U}_i & \Phi_i F \\ F^T \Phi_i^T & \bar{X} \end{bmatrix} > 0, \quad (i = 1,2,\ldots,m) \quad (26)$$

where Φ_i is a projection matrix such that $u_i = \Phi_i u$.

Furthermore, if there exists a feasible solution to the above LMIs, then the full state feedback gain $G = F\bar{X}^{-1}$.

V. FUZZY OPTIMAL VARIANCE CONTROL PROBLEM: FULL-STATE FEEDBACK

The Parallel Distributed Compensation technique, which was introduced by Wang et al. [18], is based on the T-S fuzzy model approximation. This model-based fuzzy control structure is derived according to the T-S fuzzy model, therefore each control rule is designed to compensate for each model rule of fuzzy system. Furthermore, it was shown in [18] that the convex combination of model rules can be stabilized by the same convex combination of control rules. In this section, we consider the fuzzy optimal variance control problem for Σ_{TS} (or Σ_{TSa}) by utilizing the full-state feedback controllers.

Recall the augmented system description for ith model rule Σ_{ai} as described in (9). The proposed fuzzy full-state feedback for ith control rule will have the same premise variables, i.e. "IF" statement, but different consequent, i.e. "THEN" statement. The general structure of each state feedback control rule for Σ_{ai} is given below.

Control Rule i:
IF $\theta_1(t)$ is Ξ_{i1},..., $\theta_p(t)$ is Ξ_{ip}, THEN:

$$u_a(t) = \dot{u}(t) = -F_i x_a(t), \quad (i = 1,2,\ldots,r) \quad (27)$$

where F_i is the internally stabilizing feedback gain matrix for ith T-S fuzzy model Σ_{ai}. Since (A_i, B_i) is controllable, it is straightforward to show that (A_{ai}, B_a) is also controllable. Hence, there always exists a constant gain matrix F_i so that $(A_{ai} - B_a F_i)$ is Hurwitz, for $i = 1,2,\ldots,r$. Then, the overall control input with fuzzy basis functions becomes

$$u_a(t) = -\sum_{i=1}^{r} h_i(\theta) F_i x_a(t). \quad (28)$$

Let F_i be partitioned according to $x_a(t)$ as

$$F_i = [F_{xi} \quad F_{ui}]. \quad (29)$$

Then, it follows from (28) that

$$\dot{u}(t) = -\sum_{i=1}^{r} h_i(\theta) \{F_{xi} x(t) + F_{ui} u(t)\}, \quad (30)$$

which is an ordinary differential equation in $u(t)$. Since $x(t)$ is available for feedback, $u(t)$ can be solved from (30) with given initial condition $u(0)$. Therefore, while (28) is a static full-state feedback controller for Σ_{TSa}, it is a *dynamic* full-state feedback controller for Σ_{TS}. The following theorem, which utilizes Lemmas 1 and 2, contains the main result.

Theorem 1: Consider the augmented T-S fuzzy system Σ_{TSa} described in (10). The fuzzy optimal variance control problem is solvable via a static full-state feedback controller (28), if there exist matrices S_i $(i = 1,2,\ldots,r)$ and a positive definite symmetric matrix X that minimize the output variance cost

$$\min_{X,S_1,\cdots,S_r} \text{trace}\left\{(C_{ai})_j X (C_{ai})_j^T\right\}, \quad (i = 1,2,\ldots,r;\ j = 1,2,\ldots,k_c) \quad (31)$$

subject to the following LMIs,

(i)

$$\begin{bmatrix} XA_{ai}^T + A_{ai}X - S_i^T B_a^T - B_a S_i & D_{ai} W^{\frac{1}{2}} \\ W^{\frac{1}{2}} D_{ai}^T & -I \end{bmatrix} < 0, \quad (i = 1,2,\ldots,r) \quad (32)$$

(ii)

$$\begin{bmatrix} \tau_j^2 & \Phi_j S_i \\ S_i^T \Phi_j^T & X \end{bmatrix} \geq 0, \quad (i = 1,2,\ldots,r;\ j = 1,2,\ldots,m) \quad (33)$$

(iii)

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} - \begin{bmatrix} \phi^2 I & 0 \\ 0 & 0 \end{bmatrix} \geq 0, \quad (34)$$

where X_{ij} are partitions of X according to x_a.

(iv)

$$\sigma_j^2 - \Theta_k X \Theta_k^T \geq 0, \quad (j = 1,2,\ldots,m;\ k = n+j) \quad (35)$$

where $(C_{ai})_j$ denotes the jth row of C_{ai}, and Φ_k and Θ_k are row vectors of dimension m and $n+m$, respectively, with 1 at kth entry and 0 elsewhere. Furthermore, if a feasible solution exists to the above LMIs, then the fuzzy full-state feedback gain $F_i = S_i X^{-1}$.

VI. NUMERICAL SIMULATION

In this section, we demonstrate the efficacy and performance of the proposed fuzzy full-state feedback controller by applying it to the smart airfoil model introduced in Fig. 1. Recall the equations of motion $\Sigma_{\bar{V}}$ described in (2),

$$\Sigma_{\bar{V}}: \quad \dot{x}(\tau) = A(\bar{V})x(\tau) + Bu(\tau) + Dd(\tau).$$

Before proceeding to design a T-S fuzzy state-feedback controller, we need to construct a T-S fuzzy model for $\Sigma_{\bar{V}}$. We utilize the method of *sector nonlinearity* to bound the nonlinear terms in (3) and define the nonlinear terms as follows,

$$\begin{cases} z_1(\bar{V}) = -\frac{2\bar{V}^2}{\mu} a(\bar{r}_\alpha^2 + \bar{x}_\alpha \bar{e}) + a\bar{x}_\alpha \bar{r}_\alpha^2 \\ z_2(\bar{V}) = \frac{2\bar{V}^2}{\mu} a[\bar{x}_\alpha + (1+\beta)\bar{e}] - a(1+\beta)\bar{r}_\alpha^2 \end{cases} \quad (36)$$

By choosing a range for $\bar{V} \in [\bar{V}_m \ \ \bar{V}_M]$, the minimum and maximum values of $z_1(\bar{V})$ and $z_2(\bar{V})$ can be evaluated at

$$\begin{cases} z_1(\bar{V}) = \Xi_{11}(\bar{V}) z_{1M} + \Xi_{21}(\bar{V}) z_{1m} \\ z_2(\bar{V}) = \Xi_{12}(\bar{V}) z_{2M} + \Xi_{22}(\bar{V}) z_{2m} \end{cases} \quad (37)$$

where Ξ_{ij}, $i,j=1,2$, denote the membership functions, and

$$\Xi_{1j}(z_j)=\frac{z_j-z_{jm}}{z_{jM}-z_{jm}},\quad \Xi_{2j}(z_j)=\frac{z_{jM}-z_j}{z_{jM}-z_{jm}},\quad (j=1,2) \tag{38}$$

and $z_{1M}=z_1(\bar{V}_M)$, $z_{1m}=z_1(\bar{V}_m)$, $z_{2M}=z_2(\bar{V}_M)$, $z_{2m}=z_2(\bar{V}_m)$.

Using the triangular membership functions and the model parameters listed in Table I, the T-S fuzzy models for $\Sigma_{\bar{V}}$ can be obtained as:

Model Rule 1:
IF $z_1(\bar{V})$ is "Positive" and $z_2(\bar{V})$ is "Small," THEN:
$\dot{x}(t)=A_1x+Bu+Dd$

Model Rule 2:
IF $z_1(\bar{V})$ is "Positive" and $z_2(\bar{V})$ is "Big," THEN:
$\dot{x}(t)=A_2x+Bu+Dd$

Model Rule 3:
IF $z_1(\bar{V})$ is "Negative" and $z_2(\bar{V})$ is "Small," THEN:
$\dot{x}(t)=A_3x+Bu+Dd$

Model Rule 4:
IF $z_1(\bar{V})$ is "Negative" and $z_2(\bar{V})$ is "Big," THEN:
$\dot{x}(t)=A_4x+Bu+Dd$

where the matrices $A_i,(i=1,\ldots,4)$ and $B(=D)$ are given in Table II. Therefore, the combined T-S fuzzy models which approximate $\Sigma_{\bar{V}}$ is in the form of (8).

TABLE I
THE SMART AIRFOIL MODEL PARAMETERS AND THEIR VALUES

Parameter	Value
b	$\frac{5}{12}$ (ft)
$\bar{e}$	0.35
$\bar{g}$	0.0188
$\bar{V}_m$	2.00
$\bar{V}_M$	2.98
$\bar{x}_\alpha$	0.25
$\bar{r}_\alpha$	0.6229
β	0.1
μ	152
ω_α	64.1 (rad/s)
ω_h	55.9 (rad/s)

TABLE II
THE T-S FUZZY MODEL PARAMETERS AND MATRICES

Matrix	Value
A_1	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.8959 & -0.0555 & 0.2185 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5772 & 0.0704 & -0.9757 & 0 \end{bmatrix}$
A_2	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.8959 & -0.0555 & 0.2185 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5772 & 0.0704 & -1.0933 & 0 \end{bmatrix}$
A_3	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.8959 & -0.0555 & 0.1258 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5772 & 0.0704 & -0.9757 & 0 \end{bmatrix}$
A_4	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.8959 & -0.0555 & 0.1258 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5772 & 0.0704 & -1.0933 & 0 \end{bmatrix}$
$B=D$	$\begin{bmatrix} 0 \\ -0.1428 \\ 0 \\ 0.5767 \end{bmatrix}\times 10^{-3}$

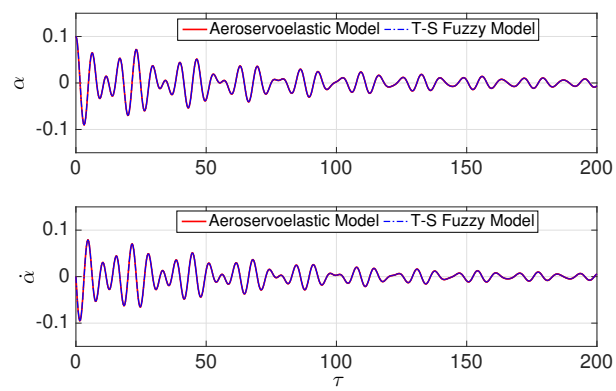

Fig. 2. T-S fuzzy model validation for α and $\dot{\alpha}$ when $\bar{V}=2.92$.

The T-S fuzzy model approximation is validated by comparing the numerical simulation between the developed T-S fuzzy model and the aeroservoelastic model (2) with given initial condition, $x_0=[0\ 0\ 0.1\ 0]$, and zero input/disturbance. For illustration, Fig. 2 shows the validation results for aerodynamic states α and $\dot{\alpha}$ when $\bar{V}=2.92$. It can be seen that the T-S model and the aeroservoelastic model are indistinguishable. Though not reported here, we have also validated T-S fuzzy models at various $\bar{V}$ within the range of consideration and they were all well matched with the corresponding aeroservoelastic models. We assume the nondimensional control input amplitude and rate bounds (3-σ) to be 1 and 300, respectively, and the random disturbance d has intensity of 0.05.

The T-S fuzzy full-state feedback control gains can be obtained by solving Eqs. (31)–(35) in Theorem 1. As an illustration, Fig. 3 shows the comparison between the open-loop and closed-loop response of aerodynamic states α and $\dot{\alpha}$ when $\bar{V}=2.92$. As can be seen, the motion of ballast mass is able to suppress the airfoil vibration effectively, and the control input and input rate are within their (3-σ) bounds, as shown in Fig. 4. The corresponding feedback gains for augmented T-S model are listed in Table III.

VII. CONCLUSIONS

In this paper, we presented a new approach to address the problem of optimal variance control for a smooth nonlinear dynamic system subject to actuator amplitude and rate constraints. The convexity property of the T-S fuzzy model was used to approximate such nonlinear system via a convex

TABLE III
T-S FUZZY FULL-STATE FEEDBACK GAINS WHEN $\bar{V} = 2.92$

Gains	Value
F_1	[-17.7546, 27.2523, -21.5733, 9.1304, 0.1694]
F_2	[-17.5771, 27.3676, -23.9292, 14.3077, 0.1713]
F_3	[-19.6901, 42.1545, -26.6340, 7.1636, 0.1474]
F_4	[-20.1295, 51.0647, -27.2346, 18.6509, 0.1358]

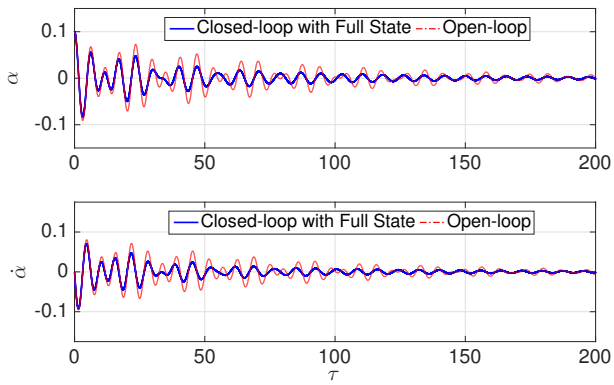

Fig. 3. Open-loop and closed-loop response comparison when $\bar{V} = 2.92$.

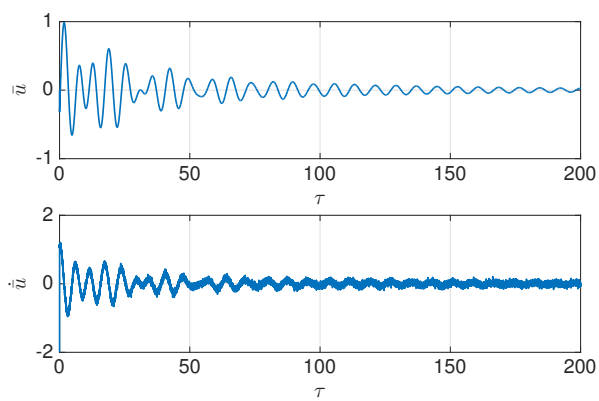

Fig. 4. Control input and input rate response when $\bar{V} = 2.92$.

combination of local linear models. The original bounded-input and bounded-input rate problem was converted into a bounded-output and bounded-input problem by augmenting the control input as a new state. This was then formulated using the LMIs by utilizing the solutions to the input and output variance constraint control problems. A fuzzy dynamic full-state feedback controller was developed based on a fuzzy static full-state feedback controller that was designed for the augmented system. This controller was then applied to controlling a smart airfoil model. The simulation results confirmed that the proposed fuzzy full-state feedback controller was effective in flutter suppression over a given range of velocity.

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