Optimal Control Modification Adaptive Law with Covariance Adaptive Gain Adjustment and Normalization

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Abstract

In the presence of large uncertainty, a controller needs to be able to adapt rapidly to regain performance. Fast adaptation is referred to the implementation of adaptive control with a large adaptive gain to reduce the tracking error rapidly. However, a large adaptive gain can lead to high-frequency oscillations which can adversely affect robustness of an adaptive control law. As the adaptive gain increases, the time delay margin for a standard model-reference adaptive control decreases, hence loss of robustness. Optimal control modification is a new adaptive control method developed recently to achieve fast adaptation with robustness. Its formulation is based on the minimization of the $\mathscr{L}_2$ norm of the tracking error, posed as an optimal control problem. Computer simulations as well as pilot-in-the-loop high-fidelity simulations in a motion-based flight simulator demonstrate the effectiveness of the new adaptive law. In this study, we extend the optimal control modification to include a covariance-like adjustment mechanism of a time-varying adaptive gain to prevent persistent learning which can reduce robustness. The covariance update law can also include a forgetting factor in a similar context as a standard recursive least-squares estimation algorithm. The covariance adaptive gain adjustment allows an initial large adaptive gain to be set arbitrarily and provides the ability to drive the adaptive gain to a lower value as the adaptation has achieved sufficiently the desired tracking performance. Alternatively, a normalized adaptive gain may be used to reduce adaptation when the amplitude of an input basis function becomes large. Flight control simulation results demonstrate that both approaches can achieve significant robustness as measured by the time delay margin. Furthermore, a recent flight test program of the optimal control modification with normalization on a NASA F-18 aircraft demonstrates the effectiveness of the adaptive law.

1 Introduction

Adaptive control has been used with success in many applications. In certain applications, fast adaptation is needed in order to improve tracking performance when a system is subject to large uncertainties such as structural damage to an aircraft that could cause rapid changes in system dynamics. In these situations, fast adaptation is needed whereby a large adaptive gain is used in an adaptive law for updating parameter estimates in order to reduce the tracking error quickly. However, there exists a balance between stability and adaptation. A large adaptive gain generally improves tracking performance. However, it is well known that a large adaptive gain for fast adaptation can result in high frequency oscillations which can excite unmodeled dynamics that could adversely affect stability of an adaptive law [1]. To address the lack of robustness of the standard model-reference adaptive control, the two well-known robust modification methods in adaptive control, namely; the σ-modification [2] and e- modification [3], have been used extensively in adaptive control.

In recent years, two new robust modification methods have been developed. The adaptive loop recovery method has been developed to provide adaptation while maintaining stability margins of the reference model [4]. The optimal control modification was developed using an optimal control framework to minimize the $\mathscr{L}_2$-norm of the tracking error

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bounded away from the origin by some lower bound [5]. By increasing the lower bound, robustness can be improved by trading off with the tracking performance.

A number of extensions have been studied with the optimal control modification method. In the presence of actuator rate limiting, a time-scale separation principle is applied to the method to decouple the slow-fast system via the singular perturbation [6]. The method improves the tracking performance in the presence of slow actuator dynamics. For problems with control input uncertainty that limits the control effectiveness, a state predictor method has been developed for the optimal control modification to accommodate both the control input uncertainty and matched uncertainty [7]. In the presence of linear matched uncertainty and for a large adaptive gain, an analytical method for estimating a lower bound of the time delay margin of the optimal control modification has been developed [7, 8].

The method has been applied to various aircraft models including an F-18 aircraft model [9] and a aeroelastic pitch dynamic model of a generic transport model [10, 11]. A high-fidelity, pilot-in-the-loop study with the optimal control modification implemented in a flight control system has recently been conducted in a motion-based flight simulator at NASA Ames. Favorable Cooper-Harper ratings by NASA test pilots have been obtained [12].

In the presence of large uncertainty, a controller needs to be able to adapt rapidly to regain the performance of a control system. Fast adaptation is referred to the implementation of adaptive control with a large adaptive gain to reduce the tracking error rapidly. However, in most situations, when an adaptation has achieved sufficiently the tracking performance, fast adaptation is usually no longer needed. Maintaining fast adaptation even after the adaptation has achieved its objective can result in persistent learning. At best, persistent learning would do nothing to further improve the tracking performance once the adaptation has achieved its objective. At worst, persistent learning reduces robustness of an adaptive controller which is not highly desired. Therefore, a new adaptive control approach in connection with the optimal control modification adaptive law has been developed to enable the adaptive gain, or learning rate, to be adjusted by a covariance-like update law [2]. The adjustment allows an arbitrarily large initial adaptive gain to be used to enable fast adaptation to reduce initial transients. The covariance adaptive gain update then adjusts the adaptive gain toward a lower value to achieve improved robustness as the adaptation proceeds. By reducing the adaptive gain, improved robustness can be achieved with the covariance adaptive gain update law, while the tracking performance during the initial adaptation is retained.

In a similar theme, normalization techniques can be used to achieve better performance of adaptive control. The objective of the normalization is not so much as to prevent persistent learning but rather to reduce the adaptation based on the amplitude of the input basis function. Both of these approaches are presented in this paper. Flight control simulation results demonstrate that both approaches can achieve significant robustness as measured by the time delay margin.

2 Optimal Control Modification with Covariance Adaptive Gain

A direct MRAC problem is posed as follows:

Given a nonlinear plant as

$$\dot{x}(t) = Ax(t) + B[u(t) + f(x(t))] \tag{1}$$

where $x(t) : [0,\infty) \to \mathbb{R}^n$ is a state vector, $u(t) : [0,\infty) \to \mathbb{R}^p$ is a control vector, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are known such that the pair (A,B) is controllable, and $f(x(t)) : \mathbb{R}^n \to \mathbb{R}^p$ is a matched uncertainty.

The uncertainty $f(x(t))$ is assumed to be linearly parametrized in the form

$$f(x(t)) = \sum_{i=1}^{n} \theta_i^* \phi_i(x(t)) + \varepsilon(x(t)) = \Theta^{*\top}\Phi(x(t)) + \varepsilon(x(t)) \tag{2}$$

where $\Theta^* \in \mathbb{R}^{m\times p}$ is an unknown constant ideal weight matrix that represents a parametric uncertainty, $\Phi(x(t)) : \mathbb{R}^n \to \mathbb{R}^m$ is a vector of known bounded basis functions that are continuous and differentiable in x, and $\varepsilon(x(t)) : \mathbb{R}^n \to \mathbb{R}^p$ is an approximation error which can be made small on a compact domain $x(t) \in \mathscr{D} \subset \mathbb{R}^n$ by a suitable selection of basis functions.

The feedback controller $u(t)$ is specified by

$$u(t) = -K_x x(t) + K_r r(t) - u_{ad}(t) \tag{3}$$

where $r(t) : [0,\infty) \to \mathbb{R}^p \in \mathscr{L}_\infty$ is a command vector, $K_x \in \mathbb{R}^{p\times n}$ is a stable gain matrix such that $A - BK_x$ is Hurwitz, $K_r \in \mathbb{R}^{p\times p}$ is a gain matrix for $r(t)$, and $u_{ad}(t) \in \mathbb{R}^p$ is a direct adaptive signal which estimates the parametric

uncertainty in the plant such that

$$u_{ad}(t) = \Theta^{\top}(t)\Phi(x(t)) \tag{4}$$

where $\Theta(t) \in \mathbb{R}^{m\times p}$ is an estimate of the parametric uncertainty Θ^*.

The reference model is specified as

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{5}$$

where $A_m \in \mathbb{R}^{n\times n}$ and $B_m \in \mathbb{R}^{n\times p}$ are given by

$$A_m = A - BK_x \tag{6}$$

$$B_m = BK_r \tag{7}$$

Let $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ be an estimation error of the parametric uncertainty and define the tracking error as $e(t) = x_m(t) - x(t)$, then the tracking error equation becomes

$$\dot{e}(t) = A_m e(t) + B\left[\tilde{\Theta}^{\top}(t)\Phi(x(t)) - \varepsilon(x(t))\right] \tag{8}$$

The parametric uncertainty can be estimated directly by the optimal control modification adaptive law as follows:

$$\dot{\Theta}(t) = -\Gamma\Phi(x(t))\left[e^{\top}(t)P - \nu\Phi^{\top}(x(t))\Theta(t)B^{\top}PA_m^{-1}\right]B \tag{9}$$

where $\nu > 0 \in \mathbb{R}$ is a weighting constant, $\Gamma = \Gamma^{\top} > 0 \in \mathbb{R}^{m\times m}$ is an adaptive gain matrix, and $P = P^{\top} > 0 \in \mathbb{R}^{n\times n}$ solves

$$PA_m + A_m^{\top}P = -Q \tag{10}$$

where $Q = Q^{\top} > 0 \in \mathbb{R}^{n\times n}$.

The derivation of the optimal control modification can be found in [5].

In this extension, the adaptive gain matrix Γ is allowed to be time-varying and is updated continuously by a covariance-like update law as in a recursive least-squares method [2]. The optimal control modification adaptive law with a covariance adaptive gain is described by

$$\dot{\Theta}(t) = -\Gamma(t)\Phi(x(t))\left[e^{\top}(t)P - \nu\Phi^{\top}(x(t))\Theta(t)B^{\top}PA_m^{-1}\right]B \tag{11}$$

$$\dot{\Gamma}(t) = \beta\Gamma(t) - \eta\Gamma(t)\Phi(x(t))\Phi^{\top}(x(t))\Gamma(t) \tag{12}$$

where $0 \le \eta < \nu\lambda_{min}\left(B^{\top}A_m^{-\top}QA_m^{-1}B\right)$ and $\beta \ge 0$ is a forgetting factor.

Theorem 1: The adaptive law (63) with the covariance adaptive gain update (64) results in stable and uniformly ultimately bounded tracking error $e(t)$ for all $\left(e(0), \tilde{\Theta}(0)\right) \in B_\alpha$ with an ultimate bound

$$\rho = \sqrt{\frac{\lambda_{max}(P)\,\delta^2 + \lambda_{max}(\Gamma^{-1})\,\kappa^2}{\lambda_{min}(P)}} \tag{13}$$

where

$$\delta = \frac{2\|PB\|\,\varepsilon_0}{\lambda_{min}(Q)} \tag{14}$$

$$\kappa = \frac{2\nu\left\|B^{\top}PA_m^{-1}B\right\|\Theta_0^*}{\nu\lambda_{min}\left(B^{\top}A_m^{-\top}QA_m^{-1}B\right) - \eta} \tag{15}$$

with $\varepsilon_0 = \sup_{x(t)\in\mathscr{D}}\|\varepsilon(x(t))\|$ and $\Theta_0^* = \max\|\Theta^*\|$.

Proof: Choose a Lyapunov candidate function

$$V(t) = e^{\top}(t)Pe(t) + \text{trace}\left(\tilde{\Theta}^{\top}(t)\Gamma^{-1}(t)\tilde{\Theta}(t)\right) \tag{16}$$

Evaluating $\dot{V}$ yields

$$\dot{V}(t) = e^{\top}(t)(A_m P + PA_m)e(t) + 2e^{\top}(t)PB\left[\tilde{\Theta}^{\top}(t)\Phi(x(t)) - \varepsilon(x(t))\right] + \text{trace}\left(-2\tilde{\Theta}^{\top}(t)\Phi(x(t))\left[e^{\top}(t)PB - \nu\Phi^{\top}(x(t))\Theta(t)B^{\top}PA_m^{-1}B\right] + \tilde{\Theta}^{\top}(t)\frac{d\Gamma^{-1}(t)}{dt}\tilde{\Theta}(t)\right) \quad (17)$$

But

$$\Gamma^{-1}(t)\Gamma(t) = I \Rightarrow \frac{d\Gamma^{-1}(t)}{dt}\Gamma(t) + \Gamma^{-1}(t)\dot{\Gamma}(t) = 0 \quad (18)$$

So

$$\frac{d\Gamma^{-1}(t)}{dt} = -\Gamma^{-1}(t)\dot{\Gamma}(t)\Gamma^{-1}(t) = -\beta\Gamma^{-1}(t) + \eta\Phi(x(t))\Phi^{\top}(x(t)) \quad (19)$$

Using the trace identity trace $\left(A^{\top}B\right) = BA^{\top}$, $\dot{V}(t)$ can be written as

$$\dot{V}(t) = -e^{\top}(t)Qe(t) + 2e^{\top}(t)PB\left[\tilde{\Theta}^{\top}(t)\Phi(x(t)) - \varepsilon(x(t))\right] - 2e^{\top}(t)PB\tilde{\Theta}^{\top}(t)\Phi(x(t)) + 2\nu\Phi^{\top}(x(t))\Theta(t)B^{\top}PA_m^{-1}B\tilde{\Theta}^{\top}(t)\Phi(x(t)) + \eta\Phi^{\top}(x(t))\tilde{\Theta}(t)\tilde{\Theta}^{\top}(t)\Phi(x(t)) - \beta\,\text{trace}\left(\tilde{\Theta}^{\top}(t)\Gamma^{-1}(t)\tilde{\Theta}(t)\right) \quad (20)$$

The sign-definiteness of the term PA_m^{-1} is now considered. Recall that a general real matrix G is positive (negative) definite if and only if its symmetric part $G = \frac{1}{2}\left(G + G^{\top}\right)$ is also positive (negative) definite. Then, by pre- and post-multiplication of Eq. (10) by $A_m^{-\top}$ and A_m^{-1}, respectively, PA_m^{-1} can be decomposed into a symmetric part M and anti-symmetric part N as

$$PA_m^{-1} = M + N \quad (21)$$

where

$$M = \frac{1}{2}\left(A_m^{-\top}P + PA_m^{-1}\right) = -\frac{1}{2}A_m^{-\top}QA_m^{-1} \quad (22)$$

$$N = \frac{1}{2}\left(PA_m^{-1} - A_m^{-\top}P\right). \quad (23)$$

Since the symmetric part $M < 0$, then $PA_m^{-1} < 0$. Thus, $\dot{V}(t)$ becomes

$$\dot{V}(t) = -e^{\top}(t)Qe(t) - 2e^{\top}(t)PB\varepsilon(x(t)) - \nu\Phi^{\top}(x(t))\tilde{\Theta}(t)B^{\top}A_m^{-\top}QA_m^{-1}B\tilde{\Theta}^{\top}(t)\Phi(x(t)) + 2\nu\Phi^{\top}(x(t))\tilde{\Theta}(t)B^{\top}NB\tilde{\Theta}^{\top}(t)\Phi(x(t)) + 2\nu\Phi^{\top}(x(t))\Theta^{*}B^{\top}PA_m^{-1}B\tilde{\Theta}^{\top}(t)\Phi(x(t)) + \eta\Phi^{\top}(x(t))\tilde{\Theta}(t)\tilde{\Theta}^{\top}(t)\Phi(x(t)) - \beta\left[V(t) - e^{\top}(t)Pe(t)\right] \quad (24)$$

Letting $y(t) = B\tilde{\Theta}^{\top}(t)\Phi(x(t))$ and using the property $y^{\top}(t)Ny(t) = 0$ for an anti-symmetric matrix N, $\dot{V}(t)$ is reduced to

$$\dot{V} = -e^{\top}(t)Qe(t) - 2e^{\top}(t)PB\varepsilon(x(t)) - \nu\Phi^{\top}(x(t))\tilde{\Theta}(t)B^{\top}A_m^{-\top}QA_m^{-1}B\tilde{\Theta}^{\top}(t)\Phi(x(t)) + 2\nu\Phi^{\top}(x(t))\Theta^{*}B^{\top}PA_m^{-1}B\tilde{\Theta}^{\top}(t)\Phi(x(t)) + \eta\Phi^{\top}(x(t))\tilde{\Theta}(t)\tilde{\Theta}^{\top}(t)\Phi(x(t)) - \beta\left[V(t) - e^{\top}(t)Pe(t)\right] \quad (25)$$

which is bounded by

$$\dot{V}(t) \leq -\|e(t)\|\left[\lambda_{min}(Q)\|e(t)\| - 2\|PB\|\varepsilon_0\right] - \|\Phi(x(t))\|^2\left\|\tilde{\Theta}(t)\right\|\left(\left[\nu\lambda_{min}\left(B^{\top}A_m^{-\top}QA_m^{-1}B\right) - \eta\right]\left\|\tilde{\Theta}(t)\right\| - 2\nu\left\|B^{\top}PA_m^{-1}B\right\|\Theta_0^{*}\right) - \beta\left[V(t) - e^{\top}(t)Pe(t)\right] \quad (26)$$

Since $V(t) - e^{\top}(t) Pe(t) > 0$, then one only needs to show that the remaining expression on the right hand side must be negative semi-definite for $\dot{V}(t) \leq 0$. Let

$$B_{\delta} = \left\{ \left(e(t), \tilde{\Theta}(t) \right) \in \mathbb{R}^{n} \times \mathbb{R}^{m \times p} : \|e(t)\| \leq \delta \text{ or } \left\| \tilde{\Theta}(t) \right\| \leq \kappa \right\} \tag{27}$$

where

$$\|e(t)\| \leq \delta = \frac{2\|PB\| \varepsilon_0}{\lambda_{min}(Q)} \tag{28}$$

$$\left\| \tilde{\Theta}(t) \right\| \leq \kappa = \frac{2\nu \left\| B^{\top} P A_m^{-1} B \right\| \Theta_0^*}{\nu \lambda_{min} \left(B^{\top} A_m^{-\top} Q A_m^{-1} B \right) - \eta} \tag{29}$$

It follows that $\dot{V} \leq 0$ for all $\left(e(t), \tilde{\Theta}(t) \right) \in B_{\Delta} - B_{\delta}$, where $B_{\Delta} = \{ e(t) \in \mathbb{R}^{n} : \|e(t)\| \leq \Delta \} \subset \mathscr{D}$. Let B_{β} be the smallest subset that encloses B_{δ}, then there exists $\beta > 0$ where

$$\beta = \lambda_{max}(P) \delta^2 + \lambda_{max}\left(\Gamma^{-1}\right) \kappa^2 \tag{30}$$

such that

$$B_{\delta} \subset B_{\beta} = \left\{ \left(e(t), \tilde{\Theta}(t) \right) \in \mathbb{R}^{n} \times \mathbb{R}^{m \times p} : V \leq \beta \right\} \tag{31}$$

Let B_{α} be the largest subset enclosed by B_{Δ}, then since $\|e(t)\| \leq \Delta$ in B_{Δ}, there exists $\alpha > 0$ where

$$\lambda_{min}(P) \|e(t)\|^2 \leq \lambda_{min}(P) \|e(t)\|^2 + \lambda_{min}\left(\Gamma^{-1}\right) \left\| \tilde{\Theta}(t) \right\|^2 \leq V \leq \lambda_{min}(P) \Delta^2 = \alpha \tag{32}$$

such that

$$B_{\alpha} = \left\{ \left(e(t), \tilde{\Theta}(t) \right) \in \mathbb{R}^{n} \times \mathbb{R}^{m \times p} :: V \leq \alpha \right\} \subset B_{\Delta} \tag{33}$$

Then for a solution to be uniformly bounded, the set containment is as follows:

$$B_{\delta} \subset B_{\beta} \subset B_{\alpha} \subset B_{\Delta} \tag{34}$$

This implies

$$\beta < \alpha \Leftrightarrow \lambda_{max}(P) \delta^2 + \lambda_{max}\left(\Gamma^{-1}\right) \kappa^2 < \lambda_{min}(P) \Delta^2 \tag{35}$$

Therefore

$$\Delta > \sqrt{\frac{\lambda_{max}(P) \delta^2 + \lambda_{max}\left(\Gamma^{-1}\right) \kappa^2}{\lambda_{min}(P)}} = \rho \tag{36}$$

where ρ is the smallest value of R.

Then ρ is the ultimate bound of $e(t)$ such that

$$\delta \leq \left\| e_p(t) \right\| \leq \rho \leq \Delta \tag{37}$$

Since $\dot{V} \leq 0$ for all $\left(e(t), \tilde{\Theta}(t) \right) \in B_{\Delta} - B_{\delta}$, therefore V is a decreasing function of time outside of B_{δ}. Thus, if $\left(e(0), \tilde{\Theta}(0) \right) \in B_{\alpha}$, the solution will eventually enters B_{β} after a finite time $t = T$ (independent of $\left(e(0), \tilde{\Theta}(0) \right)$ and α) and remain inside for all $t > T$ [13]. Therefore, $e(t)$ is uniformly ultimately bounded with an ultimate bound ρ.

Remark: Without a forgetting factor; i.e., $\beta = 0$, the adaptive gain will eventually settle to a steady-state value. To prevent a sluggishness in the adaptation due to a low value of the steady-state adaptive gain, the parameter η can be selected judiciously to maintain a reasonable value of the steady-state adaptive gain.

With a forgetting factor $\beta > 0$, the adaptive gain can grow without bound. This can lead to poor robustness and may not improve the tracking performance. Therefore, a projection method can be used to limit the value of the adaptive gain [2]. The projection method is given by:

$$\dot{\Gamma}(t) = \begin{cases} \beta \Gamma(t) - \eta \Gamma(t) \Phi(x(t)) \Phi^{\top}(x(t)) \Gamma(t) & \text{if } \|\Gamma(t)\| \leq R_0 \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

It should be mentioned that the covariance adaptive gain update can be modified with a normalization method as follows:

$$\dot{\Gamma}(t) = \beta \Gamma(t) - \eta \frac{\Gamma(t) \Phi(x(t)) \Phi^{\top}(x(t)) \Gamma(t)}{1 + \Phi^{\top}(x(t)) R \Phi(x(t))} \tag{39}$$

where $R = R^\top > 0 \in \mathbb{R}^{m\times m}$ is a normalization weight matrix.

The covariance adaptive gain update can also be used to reset the adaptation to deal with multiple sources of uncertainty that occur at different times. The resetting mechanism can be made to be triggered by a pre-defined threshold of the tracking error or other parameters. When the threshold exceeds, the adaptive gain is initialized with a large suitable value. For example, an adaptive gain resetting algorithm could be expressed as

$$\Gamma(t) = \begin{cases} \Gamma_c & \text{if } t = t_e \text{ when } \|e(t)\| > \|e(t)\|_{max} \\ \int_{t_e}^{t} \dot{\Gamma}(\tau)\, d\tau & \text{otherwise} \end{cases} \tag{40}$$

As with any limited authority control, the threshold limit should be chosen judiciously so that the trigger would occur appropriately to prevent false triggering.

3 Adaptive Gain Normalization

In lieu of the covariance adaptive gain adjustment method, normalization is a well-known technique that can be used to adjust the adaptive gain based on the amplitude of the input basis function. In a typical normalization scheme, the basis function is normalized and the Lyapunov stability proof then becomes fairly standard [2]. In the current approach, the adaptive gain is normalized instead of the basis function that results in a time-varying adaptive gain

$$\Gamma(t) = \frac{\Gamma}{1 + \Phi^\top(x(t)) R \Phi(x(t))} \tag{41}$$

The optimal control modification adaptive law with normalization is then expressed as

$$\dot{\Theta}(t) = -\frac{\Gamma}{1 + \Phi^\top(x(t)) R \Phi(x(t))} \Phi(x(t)) \left[e^\top(t) P - \nu \Phi^\top(x(t)) \Theta(t) B^\top P A_m^{-1} \right] B \tag{42}$$

The Lyapunov stability proof of the adaptive gain normalization is not quite as simple and can be stated by the following theorem:

Theorem 2: If the input basis function belongs to a class of functions such that $\|\Phi(x(t))\| \geq \|x(t)\|$ and $M\|\Phi(x(t))\| \leq \|x(t)\| \|\Phi_x(x(t))\| \leq L\|\Phi(x(t))\|$ for some constants $L > 0$ and $M > 0$, where $\Phi_x(x(t)) = d\Phi(x(t))/dx$, then the adaptive law with normalization (42) is stable.

Proof: Choose a Lyapunov candidate function

$$V(t) = \frac{1}{1 + m^2(x(t))} e^\top(t) P e(t) + \text{trace}\left(\tilde{\Theta}^\top(t) \Gamma^{-1} \tilde{\Theta}(t) \right) \tag{43}$$

where $m^2(x(t)) = \Phi^\top(x(t)) R \Phi(x(t))$.

Evaluating $\dot{V}$ yields

$$\begin{aligned} \dot{V}(t) = & \frac{1}{1 + m^2(x(t))} e^\top(t) (A_m P + P A_m) e(t) + \frac{1}{1 + m^2(x(t))} 2 e^\top(t) P B \left[\tilde{\Theta}^\top(t) \Phi(x(t)) - \varepsilon(x(t)) \right] \\ & - \frac{2m(x(t)) \dot{m}(x(t))}{[1 + m^2(x(t))]^2} e^\top(t) P e(t) - \frac{2}{1 + m^2(x(t))} \text{trace}\left(\tilde{\Theta}^\top(t) \Phi(x(t)) \left[e^\top(t) P B - \nu \Phi^\top(x(t)) \Theta(t) B^\top P A_m^{-1} B \right] \right) \end{aligned} \tag{44}$$

Using the trace identity trace $\left(A^\top B\right) = B A^\top$, $\dot{V}(t)$ can be written as

$$\begin{aligned} \dot{V}(t) = & -\frac{1}{1 + m^2(x(t))} e^\top(t) Q e(t) - \frac{2}{1 + m^2(x(t))} e^\top(t) P B \varepsilon(x(t)) - \frac{2m(x(t)) \dot{m}(x(t))}{[1 + m^2(x(t))]^2} e^\top(t) P e(t) \\ & + \frac{2\nu}{1 + m^2(x(t))} \Phi^\top(x(t)) \Theta(t) B^\top P A_m^{-1} B \tilde{\Theta}^\top(t) \Phi(x(t)) \end{aligned} \tag{45}$$

Since $m=\sqrt{\Phi^{\top}(x(t))R\Phi(x(t))}$, then the term $2m(x(t))\dot{m}(x(t))$ is evaluated as

$$2m(x(t))\dot{m}(x(t))=\dot{\Phi}^{\top}(x(t))R\Phi(x(t))+\Phi^{\top}(x(t))R\dot{\Phi}(x(t))$$
$$=\dot{x}^{\top}(t)\Phi_x^{\top}(x(t))Rx(t)+x^{\top}(t)R\Phi_x(x(t))\dot{x}(t) \quad (46)$$

This results in

$$\frac{2m(x(t))\dot{m}(x(t))}{[1+m^2(x(t))]^2}=\frac{x^{\top}(t)\left[\Phi_x^{\top}(x(t))RA_m+A_m^{\top}R\Phi_x(x(t))\right]x(t)}{[1+\Phi^{\top}(x(t))R\Phi(x(t))]^2}$$
$$+\frac{2x^{\top}(t)R\Phi_x(x(t))}{[1+\Phi^{\top}(x(t))R\Phi(x(t))]^2}\left[B_m r(t)-B\tilde{\Theta}^{\top}(t)x(t)+B\varepsilon(x(t))\right] \quad (47)$$

We will prove by counter-example by supposing that the adaptive law is unstable, thereby resulting in $x(t)\to\infty$. Then

$$\left\|-\frac{2m(x(t))\dot{m}(x(t))}{[1+m^2(x(t))]^2}\right\|\leq\frac{L^2\left\|RA_m+A_m^{\top}R\right\|\|\Phi(x(t))\|^2}{\left[1+\lambda_{min}(R)\|\Phi(x(t))\|^2\right]^2}+\frac{2L\|RB_m\|\|\Phi(x(t))\|\|r(t)\|}{\left[1+\lambda_{min}(R)\|\Phi(x(t))\|^2\right]^2}$$
$$+\frac{2L\|RB\|\|\Phi(x(t))\|^2\left\|\tilde{\Theta}(t)\right\|}{\left[1+\lambda_{min}(R)\|\Phi(x(t))\|^2\right]^2}+\frac{2\|RLB\|\|\Phi(x(t))\|\varepsilon_0}{\left[1+\lambda_{min}(R)\|\Phi(x(t))\|^2\right]^2} \quad (48)$$

For a class of functions $\Phi(x(t))$ such that $\|\Phi(x(t))\|\geq\|x(t)\|$, then as $x(t)\to\infty$, $\Phi(x(t))\to\infty$ and possibly $\left\|\tilde{\Theta}(t)\right\|\to\infty$. Since $\|r(t)\|\in\mathscr{L}_\infty$ and assuming that $\varepsilon_0\leq\|x(t)\|$, then taking the limit as $\|\Phi(x(t))\|\to\infty$ yields

$$\lim_{\|\Phi(x(t))\|\to\infty}\left\|-\frac{2m(x(t))\dot{m}(x(t))}{[1+m^2(x(t))]^2}\right\|\leq\lim_{\|\Phi(x(t))\|\to\infty}\frac{2L\|RB\|\|\Phi(x(t))\|^2\left\|\tilde{\Theta}(t)\right\|}{\left[1+\lambda_{min}(R)\|\Phi(x(t))\|^2\right]^2}$$
$$\leq\lim_{\|\Phi(x(t))\|\to\infty}\frac{2L\|RB\|\left\|\tilde{\Theta}(t)\right\|}{\lambda_{min}(R)\left[1+\lambda_{min}(R)\|\Phi(x(t))\|^2\right]} \quad (49)$$

To evaluate the right hand side, we apply the L'Hospital rule. Then

$$\lim_{\|\Phi(x(t))\|\to\infty}\frac{2L\|RB\|\left\|\tilde{\Theta}(t)\right\|}{\lambda_{min}(R)\left[1+\lambda_{min}(R)\|\Phi(x(t))\|^2\right]}=\lim_{x(t)\to\infty}\frac{2L\|RB\|\|x(t)\|^2\left\|\dot{\tilde{\Theta}}(t)\right\|}{2M^2\lambda_{min}^2(R)\|\Phi(x(t))\|^2\|\dot{x}(t)\|^2}$$
$$\leq\lim_{\|\Phi(x(t))\|\to\infty}\frac{2L\|RB\|\|\Phi(x(t))\|^2\lambda_{max}(\Gamma)\left[\|PB\|\|\Phi(x(t))\|\|e(t)\|+\nu\left\|B^{\top}PA_m^{-1}B\right\|\|\Phi(x(t))\|^2\|\Theta(t)\|\right]}{2M^2\lambda_{min}^2(R)\|\Phi(x(t))\|^2\|\dot{x}(t)\|^2\left[1+\lambda_{min}(R)\|\Phi(x(t))\|^2\right]} \quad (50)$$

Since

$$\dot{x}(t)=\dot{x}_m(t)-\dot{e}(t)=\dot{x}_m(t)-A_m e(t)-B\left[\tilde{\Theta}^{\top}(t)\Phi(x(t))-\varepsilon(x(t))\right] \quad (51)$$

Then

$$\frac{\|\Phi(x(t))\|\|e(t)\|}{\|\dot{x}(t)\|^2}\leq\frac{\|\Phi(x(t))\|\|e(t)\|}{2\|A_m\|\|B\|\|\Phi(x(t))\|\|e(t)\|\left\|\tilde{\Theta}(t)\right\|}=\frac{1}{2\|A_m\|\|B\|\left\|\tilde{\Theta}(t)\right\|} \quad (52)$$

$$\frac{\|\Phi(x(t))\|^2\|\Theta(t)\|}{\|\dot{x}(t)\|^2}\leq\frac{\|\Phi(x(t))\|^2\|\Theta(t)\|}{\|B\|^2\|\Phi(x(t))\|^2\left\|\tilde{\Theta}(t)\right\|^2}=\frac{\left\|\tilde{\Theta}(t)\right\|+\Theta_0^*}{\|B\|^2\left\|\tilde{\Theta}(t)\right\|^2} \quad (53)$$

Therefore

$$\lim_{\|\Phi(x(t))\|\to\infty}\left\|-\frac{2m\left(x\left(t\right)\right)\dot{m}\left(x\left(t\right)\right)}{\left[1+m^{2}\left(x\left(t\right)\right)\right]^{2}}\right\|$$
$$\leq\lim_{\|\Phi(x(t))\|\to\infty}\frac{2L\left\|RB\right\|\lambda_{max}\left(\Gamma\right)}{2M^{2}\lambda_{min}^{2}\left(R\right)\left[1+\lambda_{min}\left(R\right)\left\|\Phi\left(x\left(t\right)\right)\right\|^{2}\right]}\left[\frac{\left\|P\right\|}{2\left\|A_{m}\right\|\left\|\tilde{\Theta}\left(t\right)\right\|}-\frac{\nu\left\|PA_{m}^{-1}\right\|\left(\left\|\tilde{\Theta}\left(t\right)\right\|+\Theta_{0}^{*}\right)}{\left\|\tilde{\Theta}\left(t\right)\right\|^{2}}\right] \tag{54}$$

This implies that

$$\lim_{x(t)\to\infty}\frac{2m\left(x\left(t\right)\right)\dot{m}\left(x\left(t\right)\right)}{\left[1+m^{2}\left(x\left(t\right)\right)\right]^{2}}=0 \tag{55}$$

Thus

$$\dot{V}\left(t\right)=\frac{1}{1+m^{2}\left(x\left(t\right)\right)}\left[-e^{\top}\left(t\right)Qe\left(t\right)-2e^{\top}\left(t\right)PB\varepsilon\left(x\left(t\right)\right)+2\nu\Phi^{\top}\left(x\left(t\right)\right)\Theta\left(t\right)B^{\top}PA_{m}^{-1}B\tilde{\Theta}^{\top}\left(t\right)\Phi\left(x\left(t\right)\right)\right]$$
$$\leq-\frac{\left\|e\left(t\right)\right\|}{1+\lambda_{max}\left(R\right)\left\|\Phi\left(x\left(t\right)\right)\right\|^{2}}\left[\lambda_{min}\left(Q\right)\left\|e\left(t\right)\right\|-2\left\|PB\right\|\varepsilon_{0}\right]$$
$$-\frac{\left\|\Phi\left(x\left(t\right)\right)\right\|^{2}}{1+\lambda_{max}\left(R\right)\left\|\Phi\left(x\left(t\right)\right)\right\|^{2}}\left\|\tilde{\Theta}\left(t\right)\right\|\left(\nu\lambda_{min}\left(B^{\top}A_{m}^{-\top}QA_{m}^{-1}B\right)\left\|\tilde{\Theta}\left(t\right)\right\|-2\nu\left\|B^{\top}PA_{m}^{-1}B\right\|\Theta_{0}^{*}\right) \tag{56}$$

Let

$$B_{\delta}=\left\{\left(e\left(t\right),\tilde{\Theta}\left(t\right)\right)\in\mathbb{R}^{n}\times\mathbb{R}^{m\times p}:\left\|e\left(t\right)\right\|\leq\delta\text{ or }\left\|\tilde{\Theta}\left(t\right)\right\|\leq\kappa\right\} \tag{57}$$

where

$$\left\|e\left(t\right)\right\|\leq\frac{2\left\|PB\right\|\varepsilon_{0}}{\lambda_{min}\left(Q\right)} \tag{58}$$

$$\left\|\tilde{\Theta}\left(t\right)\right\|\leq\frac{2\nu\left\|B^{\top}PA_{m}^{-1}B\right\|\Theta_{0}^{*}}{\nu\lambda_{min}\left(B^{\top}A_{m}^{-\top}QA_{m}^{-1}B\right)} \tag{59}$$

Then $\dot{V}\leq 0$ for all $\left(e\left(t\right),\tilde{\Theta}\left(t\right)\right)\in B_{\Delta}-B_{\delta}$, where $B_{\Delta}=\left\{e\left(t\right)\in\mathbb{R}^{n}:\left\|e\left(t\right)\right\|\leq\Delta\right\}\subset\mathscr{D}$. The solution of $x\left(t\right)$ is thus bounded and therefore is a contradiction to the supposition that $x\left(t\right)\to\infty$. Therefore, by a counter-example argument, the adaptive law is stable and results in a bounded tracking error.

4 Flight Control Application

Consider a longitudinal pitch dynamic model of an aircraft

$$\begin{bmatrix} mV+\frac{C_{L_{\dot{\alpha}}}\bar{q}S\bar{c}}{2V} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{C_{m_{\dot{\alpha}}}\bar{q}S\bar{c}^{2}}{2V} & 0 & I_{yy} \end{bmatrix}\begin{bmatrix} \dot{\alpha}\left(t\right) \\ \dot{\theta}\left(t\right) \\ \dot{q}\left(t\right) \end{bmatrix}=\begin{bmatrix} mg\gamma-C_{L_{\alpha}}\bar{q}S & -mg\gamma & mV-\frac{C_{L_{q}}\bar{q}S\bar{c}}{2V} \\ 0 & 0 & 1 \\ C_{m_{\alpha}} & 0 & \frac{C_{m_{q}}\bar{q}S\bar{c}^{2}}{2V} \end{bmatrix}\begin{bmatrix} \alpha\left(t\right) \\ \theta\left(t\right) \\ q\left(t\right) \end{bmatrix}$$
$$+\begin{bmatrix} -C_{L_{\delta_{e}}} \\ 0 \\ C_{m_{\delta_{e}}} \end{bmatrix}\left(\delta_{e}\left(t-t_{d}\right)+\begin{bmatrix} \theta_{\alpha}^{*} & 0 & \theta_{q}^{*} \end{bmatrix}\begin{bmatrix} \alpha\left(t\right) \\ \theta\left(t\right) \\ q\left(t\right) \end{bmatrix}\right) \tag{60}$$

where $t_{d}=50$ msec is a time delay introduced to account for unmodeled dynamics.

A numerical model for a full-scale generic transport model (GTM) at Mach 0.8 and 30,000 ft with the flight path angle $\gamma = 0$ is given by

$$\begin{bmatrix} \dot{\alpha}(t) \\ \dot{\theta}(t) \\ \dot{q}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -0.7018 & 0 & 0.9761 \\ 0 & 0 & 1 \\ -2.6923 & 0 & -0.7322 \end{bmatrix}}_{A} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ q(t) \end{bmatrix} + \underbrace{\begin{bmatrix} -0.0573 \\ 0 \\ -3.5352 \end{bmatrix}}_{B} \left(\delta_e(t-t_d) + \begin{bmatrix} 0.1393 & 0 & -0.2071 \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ q(t) \end{bmatrix} \right)$$

A desired reference model of the pitch attitude is given by

$$\ddot{\theta}_m(t) + 2\zeta\omega_n\dot{\theta}_m(t) + \omega_n^2\theta_m(t) = \omega_n^2 r(t) \tag{61}$$

where $\zeta = 0.85$ and $\omega_n = 1.5$ rad/sec are chosen to give a desired handling characteristic.

Let $x(t) = \begin{bmatrix} \alpha(t) & \theta(t) & q(t) \end{bmatrix}^\top$, $u(t) = \delta_e(t)$, and $\Theta^{*\top} = \begin{bmatrix} \theta_\alpha^* & 0 & \theta_q^* \end{bmatrix} = \begin{bmatrix} 0.4 & 0 & -0.3071 \end{bmatrix}$. The parametric uncertainty Θ^* results in the short-period mode damping ratio of 0.095 which corresponds to almost neutral stability, whereas the nominal short-period mode has a damping ratio of 0.405. A nominal controller is designed as $u_{nom}(t) = -K_x x(t) + k_r r(t)$ where $K_x = \frac{1}{b_3}\begin{bmatrix} a_{31} & \omega_n^2 & 2\zeta\omega_n + a_{33} \end{bmatrix} = \begin{bmatrix} 0.7616 & -0.6365 & -0.5142 \end{bmatrix}$ and $k_r = \frac{1}{b_3}\omega_n^2 = -0.6365$. The closed-loop eigenvalues are -0.6582 and $-1.2750 \pm 0.7902i$. The nominal closed-loop plant is then chosen to be the reference model as

$$\underbrace{\begin{bmatrix} \dot{\alpha}_m(t) \\ \dot{\theta}_m(t) \\ \dot{q}_m(t) \end{bmatrix}}_{\dot{x}_m} = \underbrace{\begin{bmatrix} -0.6582 & -0.0365 & 0.9466 \\ 0 & 0 & 1 \\ 0 & -2.2500 & -2.5500 \end{bmatrix}}_{A_m} \underbrace{\begin{bmatrix} \alpha_m(t) \\ \theta_m(t) \\ q_m(t) \end{bmatrix}}_{x_m} + \underbrace{\begin{bmatrix} 0.0365 \\ 0 \\ 2.2500 \end{bmatrix}}_{B_m} r(t)$$

The actuator command is given by

$$u(t) = -K_x x(t) + k_r r(t) - \Theta^\top(t) x(t) \tag{62}$$

The optimal control modification adaptive law with the covariance adaptive gain update is specified as

$$\dot{\Theta}(t) = -\Gamma(t) x(t) \left[e^\top(t) P - \nu x^\top(t) \Theta(t) B^\top P A_m^{-1} \right] B \tag{63}$$

$$\dot{\Gamma}(t) = \beta\Gamma(t) - \eta\Gamma(t) x(t) x^\top(t) \Gamma(t) \tag{64}$$

where $e = x_m - x$.

The following parameters are chosen: $\nu = 0.2$ and $\eta = 0.4$ given that $B^\top A_m^{-\top} Q A_m^{-1} B = 2.4687$. The adaptive gain is initialized with $\Gamma(0) = 3000I$. The results of the simulation are as shown in Figures 1 to 6. For the simulation, a large adaptive gain is used to illustrate the lack of robustness of the standard MRAC in the presence of input time delay. As can be seen in Figure 2, the pitch response due to the standard MRAC exhibits large, undesired high frequency oscillations. The optimal control modification (OCM) is effective in reducing the pitch rate oscillations, although some small oscillations still exist, as shown in Figure 3. Using the covariance adaptive gain update with $\beta = 0$, Figure 4 shows that the pitch rate oscillations are almost completely eliminated. On the other hand, with a forgetting factor $\beta = 0.5$, the pitch rate response actually becomes worse as shown in Figure 5. The pitch rate response is also improved with the adaptive gain normalization as shown in Figure 6, although the tracking performance is not as good as that with the covariance adaptive gain update with $\beta = 0$.

The behaviors of the covariance adaptive gain are shown in Figures 7, 8 and 9. For $\beta = 0$, the elements of the time-varying adaptive gain matrix $\Gamma(t)$ initially start with large values, but then rapidly settle to lower values after the large high frequency initial transients in the pitch rate response, as shown in Figure 7. The elements of $\Gamma(t)$ eventually reach their steady-state values well below the initial values. The reduction in the adaptive gain values reduce the high frequency oscillations in the output signal. On the other hand, for $\beta = 0.5$, Figure 8 shows that the elements of $\Gamma(t)$

actually increase in values without bound. This behavior causes the latter part of the pitch rate response to become more oscillatory than with $\beta = 0$. Thus, $\beta = 0$ is the most appropriate setting for the covariance adaptive gain update. The projection method should used for a forgetting factor $\beta > 0$. Figure 9 is the plot of the normalized adaptive gain. The three elements of $\Gamma(t)$ collapse on top of one another since the normalization factor $1 + m^2(x(t))$ is the same for all the three adaptive gain elements.

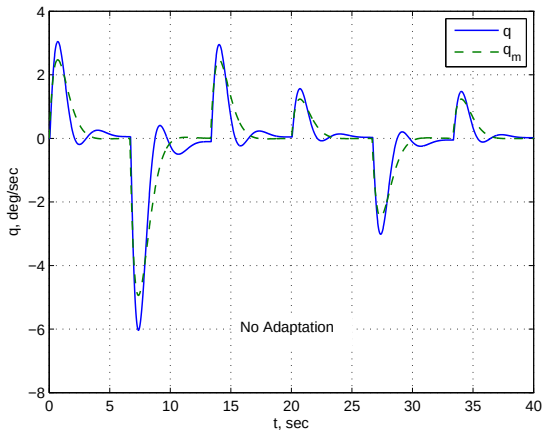

Fig. 1 - Pitch Rate Response (No Adaptation)

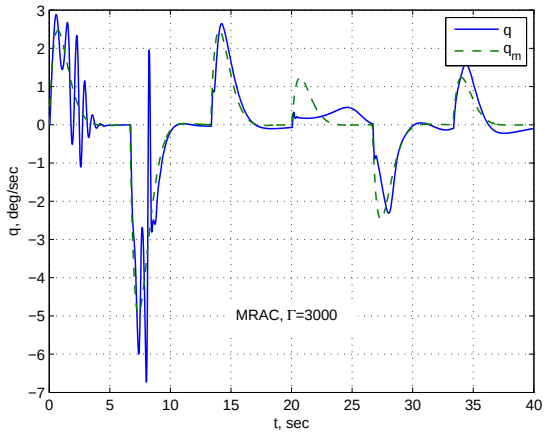

Fig. 2 - Pitch Rate Response (MRAC with $\Gamma = 3000$)

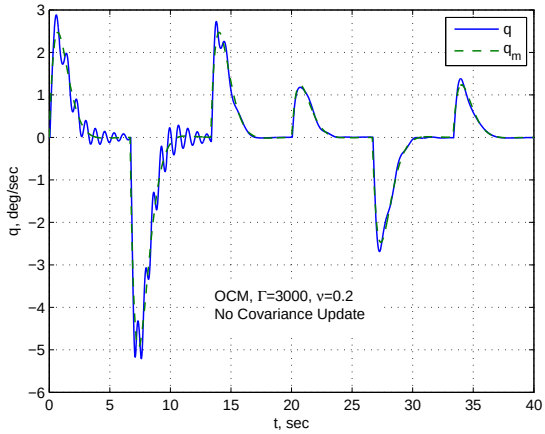

Fig. 3 - Pitch Rate Response (OCM with $\Gamma = 3000$ and $\nu = 0.2$, No Covariance Update)

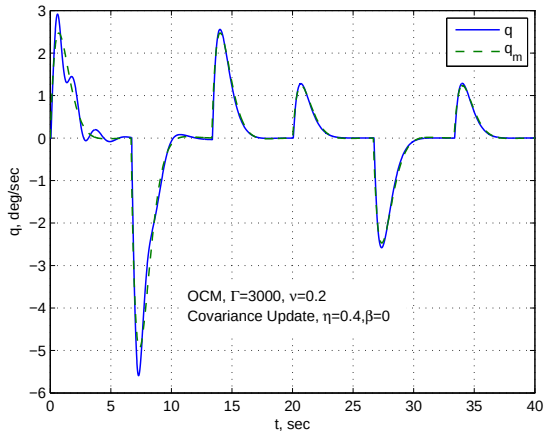

Fig. 4 - Pitch Rate Response (OCM with $\Gamma = 3000$ and $\nu = 0.2$, Covariance Update with $\eta = 0.4$ and $\beta = 0$)

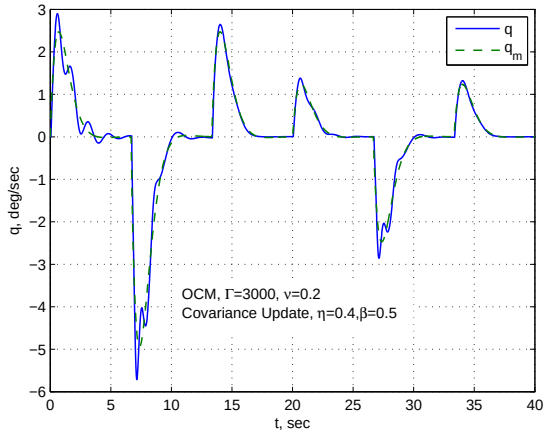

Fig. 5 - Pitch Rate Response (OCM with $\Gamma = 3000$ and $\nu = 0.2$, Covariance Update with $\eta = 0.4$ and $\beta = 0.5$)

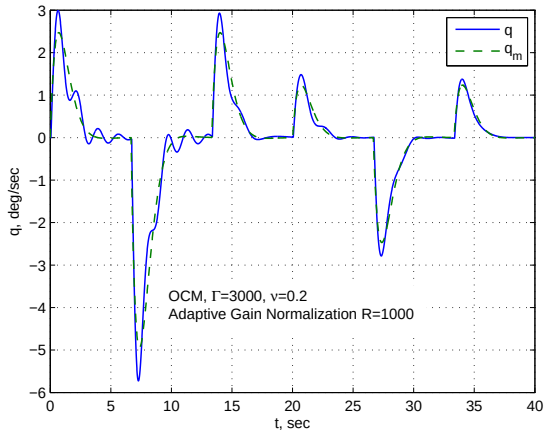

Fig. 6 - Pitch Rate Response (OCM with $\Gamma = 3000$ and $\nu = 0.2$, Adaptive Gain Normalization with $R = 1000$)

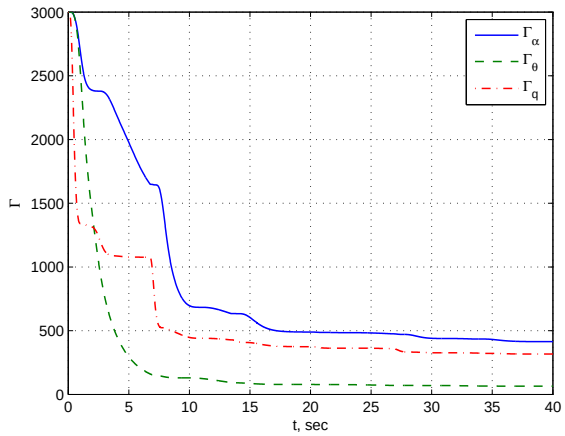

Fig. 7 - Time Varying Adaptive Gain (OCM with $\Gamma = 3000$ and $\nu = 0.2$, Covariance Update with $\eta = 0.4$ and $\beta = 0$)

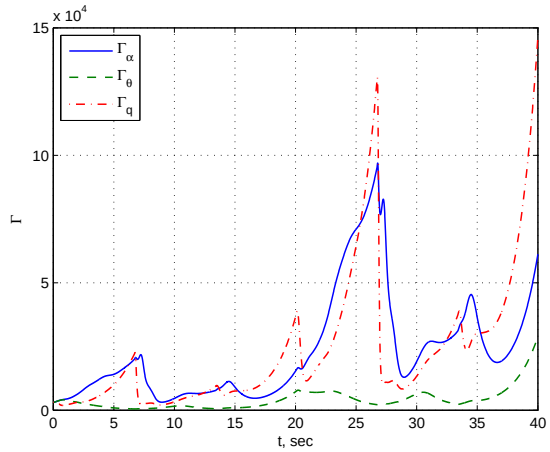

Fig. 8 - Time Varying Adaptive Gain (OCM with $\Gamma = 3000$ and $\nu = 0.2$, Covariance Update with $\eta = 0.4$ and $\beta = 0.5$)

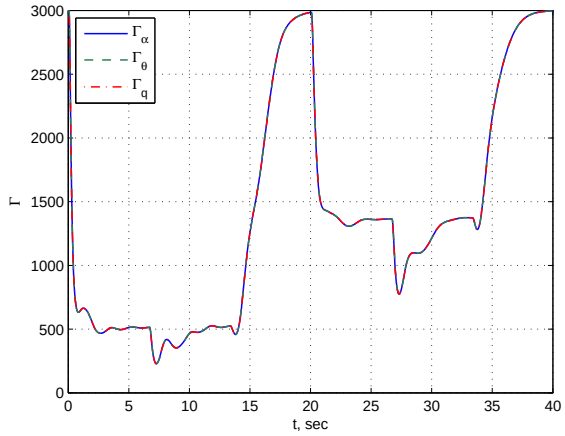

Fig. 9 - Time Varying Adaptive Gain (OCM with $\Gamma = 3000$ and $\nu = 0.2$, Adaptive Gain Normalization with $R = 1000$)

Because the covariance adaptive gain update with $\beta = 0$ adjusts the elements of $\Gamma(t)$ to lower value, high frequency oscillations are eliminated. As a result, improved robustness is obtained. Table 1 shows the time delay margin estimates of all the controller. Both the covariance adaptive gain update and with no forgetting factor ($\beta = 0$) adaptive gain normalization provide significant improvements in the time delay margin among all of the adaptive controllers.

	Time Delay Margin, msec
No Adaptation	550
MRAC	50
OCM, No Covariance Update	90
OCM, Covariance Update, $\beta = 0$	180
OCM, Covariance Update, $\beta = 0.5$	80
OCM, Normalization, $R = 1000$	190

Table 1 - Time Delay Margin Estimates

As a note, the optimal control modification adaptive law with normalization has recently been flight tested on the NASA Full-Scale Advanced Systems Testbed (FAST) which is an F/A-18A aircraft at NASA Dryden (Tail Number 853), as shown in Figure 10. The avionics include a new ARTS (Airborne Research Test System) flight control computer system. Failure emulations can be introduced via parameter changes in the flight control software to intentionally degrade aircraft stability, and through hardware emulated faults (frozen stabilator and cross-coupled pilot stick inputs) that cause a cross coupling in both the pitch and roll axes.

Fig. 10 - NASA Full-Scale Advanced Systems Testbed

Prior to the implementation in the flight control computer, the optimal control modification adaptive law was implemented in a high-fidelity flight dynamic model of the F/A-18A aircraft to evaluate the effectiveness of the adaptive law. The results show the improvement in the tracking performance of the optimal control modification adaptive law [9].

Three adaptive flight controllers were implemented in the flight control computer on board the FAST aircraft. The first adaptive flight controller is based on a simplified MRAC adaptive law (sMRAC) where the input to the adaptive law is simply a state variable. The second adaptive flight controller is based on the optimal control modification adaptive law with normalization (onMRAC). The third adaptive flight controller is a variance of the second adaptive flight controller plus an additional optimal control modification adaptive law with a bias input to handle disturbances (onMRAC+), either due to input uncertainties such as failed control surfaces or external disturbances such as coupling from another axis. This additional adaptive law with a bias input is also effectively a σ-modification adaptive law since $\Phi(x(t)) = 1$.

During a period from December of 2010 to January of 2011, a series of flight test experiments of the three adaptive flight controllers were successfully demonstrated on board the NASA FAST aircraft at NASA Dryden Flight Research Center. Initial flight test results demonstrated an obvious improvement in-flight performance with adaptive control. The "onMRAC" was noted to be working quite well and appeared to adapt more quickly than "sMRAC" for the reduced pitch damping failure. Additionally, on an early flight, the adaptive law for the disturbance estimate in the "onMRAC+" configuration was found to exhibit small, persistent oscillations in the pitch axis. Between flights, an optimal control modification term was added to this update law and shown to effectively eliminate the objectionable oscillations [14].

5 Conclusion

This study presents an extension to the optimal control modification that includes a time-varying adaptive gain which is adjusted by a covariance-like update law with a forgetting factor. The covariance adaptive gain update with no forgetting factor generally adjusts the adaptive gain to a lower value to prevent persistent learning as the adaptation proceeds. At best, persistent learning would do nothing to further improve the tracking performance once the adaptation has achieved its objective. At worst, persistent learning reduces robustness of an adaptive controller. The adjustment allows a large initial adaptive gain to be used to enable a fast adaptation to initial transients. When the adaptation has achieved sufficiently the desired tracking performance, the large initial adaptive gain is no longer needed. The covariance adaptive gain update adjusts the adaptive gain toward a lower value to achieve improved robustness. With a forgetting factor $\beta > 0$, the performance and robustness seem to degrade due to the growth in the adaptive gain. A projection method can be used to limit the value of the adaptive gain. The adaptive gain normalization is another method that can be used to improve robustness. Simulation results demonstrate the effectiveness of the covariance adaptive gain update and the adaptive gain normalization.

References

[1] Rohrs, C.E., Valavani, L., Athans, M., and Stein, G., "Robustness of Continuous-Time Adaptive Control Algorithms in the Presence of Unmodeled Dynamics," IEEE Transactions on Automatic Control, Vol AC-30, No. 9, pp. 881-889, 1985.

[2] Ioannu, P.A. and Sun, J., *Robust Adaptive Control*, Prentice-Hall, 1996.

[3] Narendra, K. S. and Annaswamy, A. M., "A New Adaptive Law for Robust Adaptation Without Persistent Excitation," IEEE Transactions on Automatic Control, Vol. AC-32**,** No. 2, pp. 134-145, 1987.

[4] Calise, A., Yucelen, T., Muse, J., Yang, B., "A Loop Recovery Method for Adaptive Control," AIAA Guidance, Navigation, and Control Conference, AIAA-2009-5967, August 2009.

[5] Nguyen, N., Krishnakumar, K., and Boskovic, J., "An Optimal Control Modification to Model-Reference Adaptive Control for Fast Adaptation", AIAA Guidance, Navigation, and Control Conference, AIAA 2008-7283, 2008.

[6] Nguyen, N., Ishihara, A., Stepanyan, V., Boskovic, J., "Singular Perturbation and Optimal Control Modification for Robust Adaptation with Slow Actuators," AIAA Guidance, Navigation, and Control Conference, AIAA-2009-5615, August 2009.

[7] Nguyen, N., "Verifiable Adaptive Control with Analytical Stability Margins by Optimal Control Modification," AIAA Guidance, Navigation, and Control Conference, AIAA-2010-7770, August 2010.

[8] Nguyen, N., "Asymptotic Linearity of Optimal Control Modification Adaptive Law with Analytical Stability Margins," AIAA Infotech@Aerospace Conference, AIAA-2010-3301, April 2010.

[9] Burken, J., Nguyen, N., Griffin, B., "Adaptive Flight Control Design with Optimal Control Modification for F- 18 Aircraft Model ," AIAA Infotech@Aerospace Conference, AIAA-2010-3364, April 2010.

[10] Tuzcu, I., Nguyen, N., "Aeroelastic Modeling and Adaptive Control of Generic Transport Model," AIAA Atmospheric Flight Mechanics, AIAA-2010-7503, August 2010.

[11] Nguyen, N., Tuzcu, I., Yucelen, T., Calise, A., "Longitudinal Dynamics and Adaptive Control Application for an Aeroelastic Generic Transport Model,", AIAA Atmospheric Flight Mechanics, August 2011.

[12] Campbell, S., Kaneshige, J., Nguyen, N., Krishnakumar, K., "An Adaptive Control Simulation Study Using Pilot Handling Qualities Evaluations," AIAA Guidance, Navigation, and Control Conference, AIAA-2010-8013, August 2010.

[13] Khalil, H. K., *Nonlinear Systems*, Prentice-Hall, 2002.

[14] Hanson, C., Schaefer, J., Burken, J., Johnson, M., Nguyen, N., "Handling Qualities Evaluations of Low Complexity Model Reference Adaptive Controllers for Reduced Pitch and Roll Damping Scenarios," AIAA Guidance, Navigation, and Control Conference, August 2011.