Robust Adaptive Model Tracking for Distributed Parameter Control of Linear Infinite-Dimensional Systems in Hilbert Space

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Abstract: This paper is focused on adaptively controlling a linear infinite-dimensional system to cause it to track a finite-dimensional reference model. Given a linear continuous-time infinite-dimensional plant on a Hilbert space and disturbances of known waveform but unknown amplitude and phase, we show that there exists a stabilizing direct model reference adaptive control law with certain disturbance rejection and robustness properties. The plant is described by a closed, densely defined linear operator that generates a continuous semigroup of bounded operators on the Hilbert space of states.

The central result will show that all errors will converge to a prescribed neighborhood of zero in an infinite dimensional Hilbert space. The result will not require the use of the standard Barbalat Lemma which requires certain signals to be uniformly continuous. This result is used to determine conditions under which a linear Infinite-dimensional system can be directly adaptively controlled to follow a reference model. In particular we examine conditions for a set of ideal trajectories to exist for the tracking problem. Our results are applied to adaptive control of general linear diffusion systems described by self-adjoint operators with compact resolvent.

I. Introduction

Many control systems are inherently infinite dimensional when they are described by partial differential equations. Currently there is renewed interest in the control of these kinds of systems especially in flexible aerospace structures, smart electric power grids, and the quantum control field [1]-[2],[18]. New general results in the theory of control of partial differential equations can be found in [11], [19]-[20]. And a very different approach to adaptive control of specifically parabolic partial differential equations can be seen in [21]. In this paper we want to consider how to make a linear infinite-dimensional system track the output of a finite-dimensional reference model in a robust fashion in the presence of persistent disturbances.

In our previous work [3]-[6] we have accomplished direct model reference adaptive control and disturbance rejection with very low order adaptive gain laws for MIMO finite dimensional systems. When systems are subjected to an unknown internal delay, these systems are also infinite dimensional in nature. Direct adaptive control theory can be modified to handle this time delay situation for infinite dimensional spaces [7]. However, this approach does not handle the situation when partial differential equations (PDEs) describe the open loop system.

This paper will provide a foundation for the topic of direct adaptive control on infinite dimensional spaces. This paper considers the effect of infinite dimensionality on the adaptive control approach of [4]-[6]. We will prove here a Robust Stability Theorem for infinite-dimensional spaces. We will show that the adaptively controlled system is robustly globally asymptotically stable using this new result. In order to accommodate robust behavior, we must give up the idea of all errors converging to zero and replace it with the idea of convergence to a prescribed neighborhood of zero whose radius is determined by the size of the unmodeled disturbance.

We want to apply this robust theory to linear PDEs governed by self-adjoint operators with compact resolvent such as linear diffusion systems. And we will also see some of the new technical difficulties encountered in infinite-dimensional direct adaptive control and find out that the devil really is in the details.

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II. Adaptive Robust Tracking with Disturbance Rejection

Let X be an infinite dimensional separable Hilbert space with *inner product* (x, y) and corresponding norm $||x|| \equiv \sqrt{(x, x)}$. Also let A be a closed linear operator with domain D(A) dense in X. Consider the *Linear Infinite Dimensional Plant with Persistent Disturbances:*

$$\begin{cases} \frac{\partial x}{\partial t}(t) = Ax(t) + Bu(t) + \Gamma u_D(t) + v, \quad x(0) \equiv x_0 \in D(A) \\ Bu \equiv \sum_{i=1}^m b_i u_i \\ y(t) = Cx(t), \quad y_i \equiv (c_i, x(t)), \quad i = 1...m \end{cases}$$
(1)

where $x \in D(A)$ is the plant state, $b_i \in D(A)$ are actuator influence functions, $c_i \in D(A)$ are sensor influence functions, $u, y \in \Re^m$ are the control input and plant output *m*-dimensional vectors respectively, and u_D is a disturbance with known basis functions ϕ_D . We assume *v* is a bounded but unknown disturbance with $\|v\| \le M_v < \infty$.

In order to accomplish some degree of disturbance rejection in a direct adaptive scheme, we will make use of a definition, given in [7], for persistent disturbances:

Definition 1: A disturbance vector $u_D \in \mathbb{R}^q$ is said to be **persistent** if it satisfies the **disturbance generator** equations:

$$\begin{cases} u_D(t) = \theta z_D(t) \\ \dot{z}_D(t) = F z_D(t) \end{cases} \quad \text{or} \quad \begin{cases} u_D(t) = \theta z_D(t) \\ z_D(t) = L \varphi_D(t) \end{cases}$$
(2)

where F is a marginally stable matrix and $\phi_D(t)$ is a vector of known functions forming a basis for all such possible disturbances. This is known as "a disturbance with known waveform but unknown amplitudes".

The *objective of control* in this paper will be to cause the output y(t) of the plant to robustly asymptotically track the output $y_m(t)$ of a linear finite-dimensional Reference Model given by:

$$\begin{cases} \dot{x}_m = A_m x_m + B_m u_m \\ y_m = C_m x_m; \quad x_m(0) = x_0^m \end{cases}$$
(3)

where the reference model state $x_m(t)$ is an N_m-dimensional vector with reference model output $y_m(t)$ having the *same* dimension as the plant output y(t). In general, the plant and reference models need **not** have the same dimensions. The excitation of the reference model is accomplished via the vector $u_m(t)$ which is generated by

$$\dot{u}_m = F_m u_m; u_m(0) = u_0^m \tag{4}$$

The reference model parameters (A_m, B_m, C_m, F_m) will be assumed completely known. What is meant by "robust asymptotic tracking" is the following: We define the **output error vector** to be

$$e_{y} \equiv y - y_{m} \xrightarrow{t \to \infty} N(0)$$
⁽⁵⁾

where N(0) is a predetermined neighborhood of the vector 0.

The control objective will be accomplished by a direct Adaptive Control Law of the form:

$$u = G_m x_m + G_u u_m + G_e e_y + G_D \varphi_D$$
(6a)

The direct adaptive controller will have adaptive gains given by:

$$\begin{cases} \dot{G}_{u} = -e_{y}u_{m}^{*}\gamma_{u}; \quad \gamma_{u} > 0\\ \dot{G}_{m} = -e_{y}x_{m}^{*}\gamma_{m}; \quad \gamma_{m} > 0\\ \dot{G}_{e} = -e_{y}e_{y}^{*}\gamma_{e}; \quad \gamma_{e} > 0\\ \dot{G}_{D} = -e_{y}\varphi_{D}^{*}\gamma_{D}; \quad \gamma_{D} > 0 \end{cases}$$
(6b)

III. Ideal Trajectories

We define the **Ideal Trajectories** for (1) in the following way:

$$\begin{cases} x_* = S_{11}^* x_m + S_{12}^* u_m + S_{13}^* z_D = S_1 z \\ u_* = S_{21}^* x_m + S_{22}^* u_m + S_{23}^* z_D = S_2 z \end{cases} \text{ with } z \equiv \begin{bmatrix} x_m \\ u_m \\ z_d \end{bmatrix} \in \mathfrak{R}^L$$
(7)

where the ideal trajectory $x_*(t)$ is generated by the ideal control $u_*(t)$ from

$$\begin{cases} \frac{\partial x_*}{\partial t} = Ax_* + Bu_* + \Gamma u_D \\ y_* = Cx_* = y_m \end{cases}$$
(8)

If such ideal trajectories exist, they will be linear combinations of the reference model state, disturbance state, and reference model input (7), and they will produce exact output tracking in a disturbance-free plant (8).

By substitution of (7) into (8) using (3)-(4), we obtain the linear **Model Matching Conditions**:

$$A S^* + B S^* - S^* A$$

$$AS_{11}^* + BS_{21}^* = S_{11}^*A_m$$
(9a)

$$AS_{12}^* + BS_{22}^* = S_{12}^*F_m + S_{11}^*B_m$$
(9b)

$$CS_{11}^* = C_m \tag{9c}$$

$$CS_{12}^* = 0 \tag{9d}$$

$$AS_{13}^* + BS_{23}^* + \Gamma\Theta = S_{13}^*F$$
(9e)

$$CS_{13}^* = 0$$
 (9f)

The Model Matching Conditions (9a)-(9f) are *necessary and sufficient* conditions for the existence of the ideal trajectories in the form of (7). These Model Matching Conditions (9a)-(9f) can be rewritten as:

$$\begin{cases} AS_1 + BS_2 = S_1L_m + H_1 \\ CS_1 = H_2 \end{cases}$$
(10)

where $S_1 \equiv \begin{bmatrix} S_{11}^* & S_{12}^* & S_{13}^* \end{bmatrix} : \mathfrak{R}^L \to D(A) \subset X$, $S_2 \equiv \begin{bmatrix} S_{21}^* & S_{22}^* & S_{23}^* \end{bmatrix} : \mathfrak{R}^L \to \mathfrak{R}^m$,

$$L_{m} \equiv \begin{bmatrix} A_{m} & B_{m} & 0\\ 0 & F_{m} & 0\\ 0 & 0 & F \end{bmatrix}, \text{ and } \begin{cases} H_{1} \equiv \begin{bmatrix} 0 & 0 & -\Gamma\theta \end{bmatrix}\\ H_{2} \equiv \begin{bmatrix} C_{m} & 0 & 0 \end{bmatrix}. \text{ Because } (S_{1}, S_{2}) \text{ are both of finite rank, they are } S_{1}, S_{2} \text{ are both of finite rank, they are } S_{2}, S_{3} \text{ are both of finite rank, they are } S_{2}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3}, S_{3} \text{ are both of finite rank, they are } S_{3} \text{ are } S_{3}, S_{3} \text{ are } S_{3} \text{ are } S_{3}, S_{3} \text{ are } S_{3} \text{ a$$

bounded linear operators on their respective domains.

IV. Ideal Trajectory Existence and Uniqueness: Normal Form

To determine conditions for the existence and uniqueness of the Ideal Trajectories, we need two lemmae: *Lemma 1:* If CB is nonsingular then $P_1 \equiv B(CB)^{-1}C$ is a (non-orthogonal) bounded projection onto the range of B, R(B), along the null space of C, N(C) with $P_2 \equiv I - P_1$ the complementary bounded projection, and $X = R(B) \oplus N(C)$ as well as $D(A) = R(B) \oplus [N(C) \cap D(A)]$.

Proof of Lemma 1: Consider

 $P_1^2 = (B(CB)^{-1}C)(B(CB)^{-1}C)$ = B(CB)^{-1}C = P_1

Hence P_1 is a projection.

Clearly, $R(P_1) \subseteq R(B)$ and $z = Bu \in R(B)$ which implies

$$P_1 z = (B(CB)^{-1}C)Bu$$
$$= Bu = z \in R(P_1)$$

Therefore $R(P_1) = R(B)$.

Also $N(P_1) = N(C)$ because $N(C) \subseteq N(P_1)$ and $z \in N(P_1)$ implies that

 $P_1 z = 0$ which implies that

$$CP_1 z = CB(CB)^{-1}Cz = 0$$
 or $N(P_1) \subseteq N(C)$

So P_2 is a projection onto R(B) along N(C) but $P_2^* \neq P_2$ so it is not an orthogonal projection in general. We have $X = R(P_1) \oplus N(P_1)$; hence $X = R(B) \oplus N(C)$.

Since $b_i \in D(A)$, we have $R(B) \subset D(A)$.

Consequently
$$D(A) = (R(B) \cap D(A)) \oplus (N(C) \cap D(A)) = R(B) \oplus (N(C) \cap D(A))$$

The projection P_1 is bounded since its range is finite dimensional, and the projection P_2 is bounded because $||P_2|| \le 1 + ||P_1|| < \infty$.

This completes the proof of Lemma 1.

Now for the above pair of projections $(P_1.P_2)$ we will have

$$\begin{cases} \frac{\partial P_1 x}{\partial t} = P_1 \frac{\partial x}{\partial t} = (\underbrace{P_1 A P_1}_{A_{11}}) P_1 x + (\underbrace{P_1 A P_2}_{A_{12}}) P_2 x + (\underbrace{P_1 B}_{B}) u \\ \frac{\partial P_2 x}{\partial t} = P_2 \frac{\partial x}{\partial t} = (\underbrace{P_2 A P_1}_{A_{21}}) P_1 x + (\underbrace{P_2 A P_2}_{A_{22}}) P_2 x + (\underbrace{P_2 B}_{=0}) u \\ y = (\underbrace{CP_1}_{C}) P_1 x + (\underbrace{CP_2}_{=0}) P_2 x \\ = 0 \end{cases}$$

which implies that

$$\begin{cases} \frac{\partial P_1 x}{\partial t} = A_{11} P_1 x + A_{12} P_2 x + B u \\ \frac{\partial P_2 x}{\partial t} = A_{21} P_1 x + A_{22} P_2 x \\ y = C P_1 x = C x \end{cases}$$

because

 $y = Cx = C(B(CB)^{-1}C)x = CP_1x ,$ $P_1x = B(CB)^{-1}Cx = B(CB)^{-1}y ,$ $CP_2 = C - CB(CB)^{-1}C = 0 , \text{ and}$ $P_2B = B - B(CB)^{-1}CB = 0 .$

Lemma 2: If CB is nonsingular, then there exists an invertible, bounded linear operator

$$W = \begin{bmatrix} C \\ W_2 P_2 \end{bmatrix} : X \to \tilde{X} \equiv R(B) x l_2 \text{ such that}$$

$$\overline{B} \equiv WB = \begin{bmatrix} CB \\ 0 \end{bmatrix} \text{ and } \overline{C} \equiv CW^{-1} = \begin{bmatrix} I_m & 0 \end{bmatrix} \text{ and } \overline{A} \equiv WAW^{-1}.$$

This coordinate transformation can be used to put (1) into normal form:

$$\begin{cases} \dot{y} = \overline{A}_{11}y + \overline{A}_{12}z_2 + CBu \\ \frac{\partial z_2}{\partial t} = \overline{A}_{21}y + \overline{A}_{22}z_2 \end{cases}$$
(11)

where the subsystem: $(\overline{A}_{22}, \overline{A}_{12}, \overline{A}_{21})$ is called the *zero dynamics* of (1) and $\overline{A}_{11} \equiv CA_{11}B(CB)^{-1} = CAB(CB)^{-1}; \overline{A}_{12} \equiv CAW_2^*; \ \overline{A}_{21} \equiv W_2A_{21}B(CB)^{-1}; \overline{A}_{22} \equiv W_2A_{22}W_2^*$ and $W_2: X \rightarrow l_2$ by $W_2 x \equiv \begin{bmatrix} (\theta_1, P_2 x) \\ (\theta_2, P_2 x) \\ (\theta_3, P_2 x) \\ \dots \end{bmatrix}$ is an isometry from N(C) into l_2 .

Proof of Lemma 2:

Since *X* is separable, we can let $N(C) = \overline{sp} \{\theta_k\}_{k=1}^{\infty}$ be an orthonormal basis. $\left[(\theta_k, P_k x) \right]$

Define
$$W_2: X \to l_2$$
 by $W_2 x \equiv \begin{bmatrix} (1, 1, 2) \\ (\theta_2, P_2 x) \\ (\theta_3, P_2 x) \\ \dots \end{bmatrix}$.

Note that $||W_2 x||^2 = \sum_{k=1}^{\infty} |(\theta_k, P_2 x)|^2 = ||P_2 x||^2 < \infty$ which implies $W_2 x \in l_2$.

So W_2 is a bounded linear operator, and an isometry of $W_2N(C)$ into l_2 .

Consequently $W_2 W_2^* = I$ on N(C).

Then we have $W_2^*W_2 = P_2$ and the retraction: $z_2 = W_2 P_2 x \in l_2$.

Also
$$W_2^* z_2 = W_2^* (W_2 P_2 x) = P_2 x$$
.
Now, using $x = P_1 x + P_2 x$ from lemma 1, we have
 $\dot{y} = CP_1 \dot{x}$
 $= CP_1 A(P_1 x + P_2 x) + CP_1 B u$
 $= C(B(CB)^{-1}C)AB(CB)^{-1} y + C(B(CB)^{-1}C)A(W_2^* z_2) + C(B(CB)^{-1}C)B u$
 $= \overline{A}_{11} y + \overline{A}_{12} z_2 + CB u$
and
 $\dot{z}_2 = W_2 P_2 \dot{x}$
 $= WP_2 [A(P_1 x + P_2 x) + B u]$
 $= W_2 P_2 A(B(CB)^{-1} y + W_2^8 z_2) + W_2 P_2 B u$
 $= W_2 (I - B(CB)^{-1}B)AB(CB)^{-1} y + W_2 (I - B(CB)^{-1}B)AW_2^* z_2$
 $= \overline{A}_{21} y + \overline{A}_{22} z_2$.
This yields the normal form (11).

Choose $W \equiv \begin{bmatrix} C \\ W_2 P_2 \end{bmatrix}$ which is a bounded linear operator. Then W has a bounded inverse explicitly stated as

$$W^{-1} \equiv \begin{bmatrix} B(CB)^{-1} & W_2^* \end{bmatrix}.$$

This gives

$$WW^{-1} = \begin{bmatrix} CB(CB)^{-1} & CW_2^* \\ W_2P_2B(CB)^{-1} & W_2P_2W_2^* \end{bmatrix}$$
$$= \begin{bmatrix} I_m & 0 \\ 0 & W_2W_2^* \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I \end{bmatrix} = I$$

because $R(W_2^*) \subseteq N(C)$.

Furthermore, $W^{-1}W = P_1 + W_2^*W_2P_2 = P_1 + P_2 = I$ because $W_2W_2^* = I$ on N(C). Also direct calculation yields:

$$\begin{cases} \overline{B} \equiv WB = \begin{bmatrix} CB \\ W_2 P_2 B \end{bmatrix} = \begin{bmatrix} CB \\ 0 \end{bmatrix} \\ \overline{C} \equiv CW^{-1} = \begin{bmatrix} CB(CB)^{-1} & CW_2^* \end{bmatrix} = \begin{bmatrix} I_m & 0 \end{bmatrix} \\ \overline{A} \equiv WAW^{-1} = \begin{bmatrix} CAB(CB)^{-1} & CAW_2^* \\ W_2 P_2 AB(CB)^{-1} & W_2 P_2 A P_2 W_2^* \end{bmatrix}$$

This completes the proof of Lemma 2.

Now we can prove the following theorem about the Existence and Uniqueness of Ideal Trajectories: **Theorem 1:** Assume *CB* is nonsingular. Then $\sigma(L_m) = \sigma(A_m) \cup \sigma(F_m) \cup \sigma(F) \subset \rho(\overline{A}_{22})$ where $\rho(\overline{A}_{22}) \equiv \{\lambda \in C \text{ such that } (\lambda I - \overline{A}_{22})^{-1} : l_2 \rightarrow l_2 \text{ is a bounded linear operator} \}$ if and only if there exist unique bounded linear operator solutions (S_1, S_2) satisfying the Matching Conditions (10). Note that we can also write $\sigma(L_m) \cap \sigma(\overline{A}_{22}) = \varphi$ where $\sigma(\overline{A}_{22}) \equiv [\rho(\overline{A}_{22})]^c$.

Proof of Theorem 1: Define $\overline{S}_1 \equiv W^{-1}S_1 = \begin{bmatrix} \overline{S}_a \\ \overline{S}_b \end{bmatrix}$ and $\overline{H}_1 \equiv WH_1 = \begin{bmatrix} \overline{H}_a \\ \overline{H}_b \end{bmatrix}$.

From (10), we obtain

$$\begin{cases} \overline{A}\overline{S}_1 + \overline{B}S_2 = \overline{S}_1L_m + \overline{H}_1 \\ \overline{C}\overline{S}_1 = H_2 \end{cases}$$

where $(\overline{A}, \overline{B}, \overline{C})$ is the Normal Form (11).

From this we obtain:

$$\begin{cases}
\overline{S}_a = H_2 \\
S_2 = (CB)^{-1} [H_2 L_m + \overline{H}_a - (\overline{A}_{11} H_2 + \overline{A}_{12} \overline{S}_b)]. \\
\overline{A}_{22} \overline{S}_b - \overline{S}_b L_m = \overline{H}_b - \overline{A}_{21} H_2
\end{cases}$$

We can rewrite the last of these equations as $(\lambda I - \overline{A}_{22})\overline{S}_b - \overline{S}_b(\lambda I - L_m) = \overline{A}_{21}H_2 - \overline{H}_b \equiv \overline{H}$ for all complex λ .

Now assume that L_m is simple and therefore provides a basis of eigenvectors $\{\phi_k\}_{k=1}^L$ for \Re^L . This is not essential but will make this part of the proof easier to understand. The proof can be done with generalized eigenvectors and the Jordan form. So we have

$$(\lambda_k I - \overline{A}_{22})\overline{S}_b \varphi_k - \overline{S}_b \underbrace{(\lambda_k I - L_m)\varphi_k}_{=0} = \overline{A}_{21}H_2 - \overline{H}_b \equiv \overline{H}$$

which implies that

$$\overline{S}_{b}\varphi_{k} = (\lambda_{k}I - \overline{A}_{22})^{-1}\overline{H}\varphi_{k}$$

because $\lambda_{k} \in \sigma(L_{m}) \subset \rho(\overline{A}_{22})$.

Thus we have

$$\overline{S}_b z = \sum_{k=1}^L \alpha_k (\lambda_k I - \overline{A}_{22})^{-1} \overline{H} \phi_k \forall z = \sum_{k=1}^L \alpha_k \phi_k \in \mathfrak{R}^L.$$

Since $\lambda_k \in \sigma(L_m) \subset \rho(\overline{A}_{22})$, all $(\lambda_k I - \overline{A}_{22})^{-1}$ are bounded operators. Also $\overline{H} \equiv \overline{A}_{21}H_2 - \overline{H}_b$ is a bounded operator on \Re^L .

Therefore \overline{S}_b is a bounded linear operator, and this leads to S_1 also bounded linear. If we look at the converse statement and let $\lambda_* \in \sigma(L_m) \cap \sigma(\overline{A}_{22}) = \phi$.

Then there exists $\varphi_* \neq 0$ such that

$$(\lambda_*I - \overline{A}_{22})\overline{S}_b\varphi_* - \overline{S}_b\underbrace{(\lambda_*I - L_m)\varphi_*}_{=0} = (\lambda_*I - \overline{A}_{22})\overline{S}_b\varphi_* = \overline{H}.$$

In this case *three* things can happen when $\lambda_* \in \sigma(\overline{A}_{22})$:

- 1) $(\lambda_* I \overline{A}_{22})$ can fail to be one to one so multiple solutions of \overline{S}_b will exist
- 2) $R(\lambda_* I \overline{A}_{22})$ can fail to be all of X so no solutions \overline{S}_b may occur or
- 3) $(\lambda_* I \overline{A}_{22})^{-1}$ can fail to be a bounded operator so solutions \overline{S}_b may be unbounded.

In all cases these three alternatives lead to a lack of unique bounded operator solutions for S_1 . The proof of Theorem 1 is complete.

It is possible to relate the point spectrum $\sigma_p(\overline{A}_{22}) \equiv \{\lambda \text{ such that } \lambda I - \overline{A}_{22} \text{ is not one to one}\}$ to the set Z of *transmission (or blocking) zeros* of (A, B, C). As in the finite-dimensional case [16], we can see that

$$Z = \left\{ \lambda \text{ such that } V(\lambda) = \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} : D(A) x \mathfrak{R}^m \to X x \mathfrak{R}^m \text{ linear operator is not one to one} \right\}$$

Lemma 3: $Z = \sigma_p(\overline{A}_{22}) = \{\lambda \text{ such that } \lambda I - \overline{A}_{22} \text{ is not one to one}\}$ is called the *point spectrum* of \overline{A}_{22} .

So the transmission zeros of the infinite-dimensional open-loop plant (A, B, C) are the eigenvalues of its zero dynamics $(\overline{A}_{22}, \overline{A}_{12}, \overline{A}_{21})$.

Proof of Lemma 3:

From
$$\overline{V}(\lambda) = \begin{bmatrix} \lambda I - \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}$$
 we obtain

$$\begin{bmatrix} \lambda I - \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix}$$
 not one to one if and only if
$$\begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix}$$
 is not one to one.
But, using the Normal Form from Lemma 2,

$$\overline{V}(\lambda) = \begin{bmatrix} \lambda I - \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \begin{bmatrix} \lambda I - \overline{A}_{11} & -\overline{A}_{12} & CB \\ -\overline{A}_{21} & \lambda I - \overline{A}_{22} & 0 \\ I_m & 0 & 0 \end{bmatrix}$$

And therefore $0 = \overline{V}(\lambda)h = \overline{V}(\lambda) \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$ if and only if
 $h_1 = 0, h_3 = (CB)^{-1}\overline{A}_{12}h_2$, and $(\lambda I - \overline{A}_{22})h_2 = 0$.
So $h \neq 0$ if and only if $h_2 \neq 0$.
Therefore $\begin{bmatrix} sI - \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix}$ is not one to one if and only if $\lambda \in \sigma_p$ (\overline{A}_{22}) .

This completes the proof of Lemma 3.

Using Lemma 3 and Theo. 1, we have the following Internal Model Principle:

Corollary 1: Assume *CB* is nonsingular and

$$\sigma(\overline{A}_{22}) = \sigma_p(\overline{A}_{22}) = \sigma_p(P_2AP_2)$$
 where $\overline{A}_{22} \equiv W_2^*P_2AP_2W_2$

There exist unique bounded linear operator solutions (S_1, S_2) satisfying the Matching Conditions (10) if and only if $\sigma(L_m) \cap Z = [\sigma(A_m) \cup \sigma(F_m) \cup \sigma(F)] \cap Z = \varphi$, i.e., no eigenvalues of (A_m, F_m, F) can be zeros of (A, B, C).

Note:

$$\lambda I - \overline{A}_{22} \text{ is not } 1 - 1 \Leftrightarrow \exists x \neq 0 \Rightarrow P_2 x \neq 0 \& z_2 = W_2 P_2 x \neq 0 \& (\lambda I - \overline{A}_{22}) z_2 = 0$$

$$\Leftrightarrow \exists x \neq 0 \Rightarrow P_2 x \neq 0 \& 0 = (\lambda I - \overline{A}_{22}) W_2 P_2 x = (\lambda \underbrace{W_2 W_2^*}_{I} - W_2 PA P_2 W_2^*) W_2 P_2 x$$

 $= [W_2(\lambda I - P_2AP_2)W_2^*]W_2P_2x$ $\Leftrightarrow W_2(\lambda I - P_2AP_2)W_2^* \text{ is not } 1 - 1 \text{ on } N(C)$ But W_2 is an isometry on N(C)

$$\therefore \sigma_p(\overline{A}_{22}) = \sigma_p(P_2AP_2).$$

V. Stability of the Error System: Almost Strict Dissipativity

The error system can be found from (1) and (8) by first defining

 $e \equiv x - x_*$ and $\Delta u \equiv u - u_*$. Then we have

$$\begin{cases} \frac{\partial e}{\partial t} = Ae + B\Delta u + v \\ e_y \equiv y - y_m = y - y_* = Ce \end{cases}$$
(12)

Now consider the definition of Strict Dissipativity for infinite-dimensional systems and the general form of this **adaptive error system** to prove stability. The main theorem of this section will later be utilized to assess the convergence and stability of the adaptive controller with disturbance rejection for linear diffusion systems.

Noting that there can be some ambiguity in the literature with the definition of strictly dissipative systems, we modify the suggestion of Wen in [8] for finite dimensional systems and expand it to include infinite dimensional systems.

Definition 2: The triple (A_c, B, C) is said to be **Strictly Dissipative** (**SD**) if A_c is a densely defined ,closed operator on $D(A_c) \subseteq X$ a complex Hilbert space with inner product (x, y) and corresponding norm $||x|| \equiv \sqrt{(x, x)}$ and generates a C_0 semigroup of bounded operators U(t), and (B, C) are bounded finite

rank input/output operators with rank M where $B: \mathbb{R}^m \to X$ and $C: X \to \mathbb{R}^m$. In addition there exist symmetric positive bounded operators P and Q on X such that

 $0 \le p_{\min} \left\| e \right\|^2 \le (Pe, e) \le p_{\max} \left\| e \right\|^2; \quad 0 \le q_{\min} \left\| e \right\|^2 \le (Qe, e) \le q_{\max} \left\| e \right\|^2, \text{ i.e. } P \text{ and } Q \text{ are bounded and coercive, and}$

$$\begin{cases} \operatorname{Re}(PA_{c}e,e) \equiv \frac{1}{2} \Big[(PA_{c}e,e) + \overline{(PA_{c}e,e)} \Big] = \frac{1}{2} \Big[(PA_{c}e,e) + (e,PA_{c}e) \Big] \\ = -(Qe,e) \leq -q_{\min} \|e\|^{2}; \quad e \in D(A_{c}) \\ PB = C^{*} \end{cases}$$
(13)

where C^* is the adjoint of the operator C.

We also say that (A, B, C) is Almost Strictly Dissipative (ASD) when there exists a $G_* \in \Re^{mxm}$ such that (A_c, B, C) is strictly dissipative with $A_c \equiv A + BG_*C$.

Note that if P = I in (13), by the Lumer-Phillips Theorem [10], p405, we would have $||U_c(t)|| \le e^{-\sigma t}; t \ge 0; \sigma \equiv q_{\min} > 0.$

Henceforth, we will make the following set of assumptions: *Hypothesis 1:* Assume the following:

- i.) There exists a gain, G_e^* such that the triple $(A_C \equiv A + BG_e^*C, B, C)$ is strictly dissipative, i.e. (A, B, C) is ASD,
- ii.) A is a densely defined , closed operator on $D(A) \subseteq X$ and generates a C_0 semigroup of bounded operators U(t), and
- iii.) φ_D is bounded

From (7), we have $u_* = S_{21}^* x_m + S_{22}^* u_m + S_{23}^* z_D$ and using (6), we obtain:

$$\Delta u \equiv u - u_{*} = (G_{m}x_{m} + G_{u}u_{m} + G_{e}e_{y} + G_{D}\varphi_{D}) - (S_{21}^{*}x_{m} + S_{22}^{*}u_{m} + S_{23}^{*}z_{D})_{L\varphi_{D}}$$

$$= G_{e}^{*}e_{y} + \Delta G_{e}e_{y} + [\Delta G_{m} \quad \Delta G_{u} \quad \Delta G_{D}]\begin{bmatrix}x_{m}\\u_{m}\\\varphi_{D}\end{bmatrix} = G_{e}^{*}e_{y} + \Delta G\eta$$

where $\Delta G \equiv G - G_*; G \equiv \begin{bmatrix} G_e & G_m & G_u & G_D \end{bmatrix}; G_* \equiv \begin{bmatrix} G_e^* & S_{21}^* & S_{22}^* & S_{23}^*L \end{bmatrix};$ and $\eta \equiv \begin{bmatrix} e_y & x_m & u_m & \varphi_D \end{bmatrix}^T$ From (1), (6), (12), and (13), the *Error System* becomes

$$\begin{vmatrix} \frac{\partial e}{\partial t} = (\underbrace{A + BG_e^*C}_{A_c})e + B\Delta G\eta + v = A_c e + B\rho + v; e \in D(A); \rho \equiv \Delta G\eta \\ e_y = Ce \\ \Delta \dot{G} = \dot{G} - \dot{G}_* = \dot{G} = -e_y \eta^* \gamma; \quad \gamma \equiv \begin{bmatrix} \gamma_e & 0 & 0 & 0 \\ 0 & \gamma_m & 0 & 0 \\ 0 & 0 & \gamma_u & 0 \\ 0 & 0 & 0 & \gamma_D \end{bmatrix} > 0$$
(15)

Since *B*, *C* are finite rank operators, so is BG_e^*C . Therefore, $A_c \equiv A + BG_e^*C$ with $D(A_c) = D(A)$ generates a C_0 semigroup $U_c(t)$ because *A* does; see [9] Theo. 2.1 p. 497. Furthermore, by Theorem 8.10 p 157 in [11], x(t) remains in D(A) and is differentiable there for all $t \ge 0$. This is because $F(t) \equiv B\rho = B\Delta G\eta$ is continuously differentiable in D(A).

We see that (14) is the *feedback interconnection* of an infinite-dimensional linear subsystem with $e \in D(A) \subseteq X$ and a finite-dimensional subsystem with $\Delta G \in \Re^{mxm}$. This can be written in the following form using $w \equiv \begin{bmatrix} e \\ \Delta G \end{bmatrix} \in D \equiv D(A)x \Re^{mxm} \subseteq \overline{X} \equiv Xx \Re^{mxm}$:

$$\begin{cases} \frac{\partial w}{\partial t} = w_t = f(t, w) \equiv \begin{bmatrix} A_c e + B\rho(t) + v \\ -e_y \eta^* \gamma \end{bmatrix} \\ w(t_0) = w_0 \in D \text{ dense in } \overline{X} \equiv X \mathfrak{M}^{mxm} \end{cases}$$
(16)

The inner product on $\overline{X} \equiv X \mathfrak{R}^{m \times m}$ can be defined as

$$(w_1, w_2) \equiv \left(\begin{bmatrix} x_1 \\ \Delta G_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ \Delta G_2 \end{bmatrix} \right)$$
 which will make it a Hilbert space also.
$$\equiv (x_1, x_2) + \operatorname{trace}(\Delta G_2 \Delta G_1^*)$$

The following Robust Stabilization Theorem shows that convergence to a neighborhood with radius determined by the supremum norm of ν is possible for a specific type of adaptive error system. In the following, we denote $\|M\|_{\gamma} \equiv \sqrt{\operatorname{tr}(M\gamma^{-1}M^T)}$ as the trace norm of a matrix M where $\gamma > 0$. Theorem 2 (Robust Stabilization): Consider the coupled system of differential equations

$$\frac{\partial e}{\partial t} = A_c e + B \underbrace{\left(G(t) - G^*\right)}_{\Delta G} z + \nu$$

$$e_y = Ce$$

$$\dot{G}(t) = -e_y z^{\mathrm{T}} \gamma - aG(t)$$
(17)

where $e, v \in D(A_c), z \in \mathbb{R}^m$ and $\begin{bmatrix} e \\ G \end{bmatrix} \in \overline{X} \equiv X \times \mathbb{R}^{m \times m}$ is a Hilbert space with inner product $\begin{pmatrix} e_1 \\ G_1 \end{pmatrix}, \begin{bmatrix} e_2 \\ G_2 \end{pmatrix} \equiv (e_1, e_2) + \operatorname{tr}(G_1 \gamma^{-1} G_2), \text{ norm } \begin{bmatrix} e \\ G \end{bmatrix} \equiv (\|e\|^2 + \operatorname{tr}(G \gamma^{-1} G))^{\frac{1}{2}}$ and where

G(t) is the mxm adaptive gain matrix and γ is any positive definite constant matrix, each of appropriate dimension. Assume the following:

- i.) (A, B, C) is ASD with $A_c \equiv A + BG_*C$
- ii.) there exists $M_G > 0$ such that $\sqrt{\operatorname{tr}(G^* G^{*T})} \leq M_G$
- iii.) there exists $M_{\nu} > 0$ such that $\sup_{t \ge 0} ||v(t)|| \le M_{\nu} < \infty$

iv.) there exists $\alpha > 0$ such that $a \le \frac{q_{\min}}{p_{\max}}$, where q_{\min} , p_{\max} are defined in Definition 2

v.) the positive definite matrix γ satisfies $\operatorname{tr}(\gamma^{-1}) \leq \left(\frac{M_{\nu}}{aM_{\alpha}}\right)^{2}$,

then the gain matrix, G(t), is bounded, and the state, e(t) exponentially with rate e^{-at} approaches the ball of radius

$$R_* \equiv \frac{\left(1 + \sqrt{p_{\max}}\right)}{a\sqrt{p_{\min}}} M_{\nu}$$

Proof of Theorem 2: See Appendix I.

Now we can prove the robust stability and convergence of the direct adaptive controller (4) in closed-loop with the linear infinite-dimensional plant (1)-(2).

Theorem 3: Under Hypothesis 1, we have robust state and output tracking of the reference model:

 $\left| \begin{array}{c} C \\ \Delta G \end{array} \right| \xrightarrow{t \to \infty} N(0, R_*)$ and since *C* is a bounded linear operator, we have:

 $e_y = y - y_m = Ce \xrightarrow{t \to \infty} N(0, R_*)$ with bounded adaptive gains $G \equiv \begin{bmatrix} G_e & G_m & G_u & G_D \end{bmatrix} = G_* + \Delta G$

Proof of Theo. 3: Follows directly from application of Theo2 to the error system (12) or (17). Note that uniform continuity is not needed since Barbalat's Lemma [15] is not invoked here.

VI. Application: Adaptive Control of Unstable Diffusion Equations Described by Self Adjoint Operators with Compact Resolvent

We will apply the above direct adaptive controller on the following single-input/single-output Cauchy problem:

$$\begin{cases} \frac{\partial x}{\partial t} = Ax + b(u + u_D) + v, x(0) \equiv x_0 \in D(A) \\ y = (c, x), \text{ with } b, c \in D(A) \end{cases}$$

And the reference model will be

$$\begin{cases} \dot{x}_m = A_m x_m + B_m u_m = -x_m + u_m \\ y_m = C_m x_m = x_m \\ \dot{u}_m = F_m u_m = 0 \end{cases}$$

For this application we will assume the disturbances are step functions. Note that the disturbance functions can be any basis function as long as φ_D is bounded, in particular sinusoidal disturbances are often applicable. So

we have
$$\varphi_D \equiv 1$$
 and $\begin{cases} u_D = (1)z_D \\ \dot{z}_D = (0)z_D \end{cases}$ which implies $F = 0$ and $\theta_D = 1$.
Let $u = G_e y + G_D$ with $\begin{cases} \dot{G}_e = -e_y e_y^* \gamma_e \\ \dot{G}_D = -e_y \gamma_D \end{cases}$.

We will assume that A is closed and densely defined, but is also a *self adjoint operator with compact resolvent*. This means A has discrete real spectrum: $\lambda_1 \ge \lambda_2 \ge \dots \longrightarrow -\infty$ and $\{\varphi_k\}_{k=1}^{\infty}$ an orthonormal sequence of eigenfunctions; see [9] Theo 6.29 p187.

Assume $\lambda_k \neq 0 \forall k = 1, 2, \dots$

Only a finite number of the eigenvalues maybe unstable (or positive); so we will say:

 $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N \ge -\sigma \ge \lambda_{N+1} \to -\infty$, where $\sigma > 0$ is the desired stability margin.

Define the **Orthogonal Projection Operators**: $x = P_N x + P_R x$

with
$$P_N \equiv \sum_{k=1}^N \underbrace{(x, \varphi_k)}_{x_k} \varphi_k, P_R \equiv \sum_{k=N+1}^\infty \underbrace{(x, \varphi_k)}_{x_k} \varphi_k$$
 where $P_N : X \to S_N \equiv sp\{\varphi_1, ..., \varphi_N\}, P_R : X \to S_N^{\perp}$.

Let the sensor and actuator influence functions be the same and entirely in S_N :

$$c \equiv b = \sum_{k=1}^{N} (b, \phi_k) \phi_k$$
 with all $b_k \neq 0$ and choose $G_* \equiv -g^* < 0$. Then $A_c = A - g_* b^* b$ remains self-

adjoint with discrete spectrum,

and
$$\begin{cases} A_c P_N x = \sum_{k=1}^N \lambda_k P_N \varphi_k - g_* P_N b^* b x = \sum_{k=1}^N \lambda_k \varphi_k - g_* (b, P_N x) b \\ A_c P_R x = \sum_{k=N+1}^\infty \lambda_k P_R \varphi_k = \sum_{k=N+1}^\infty \lambda_k \varphi_k \end{cases}$$
 because $P_N b = b = c$.

So $\operatorname{Re}(PA_c x, x) = \operatorname{Re}(PA_c P_N x, x) + \operatorname{Re}(PA_c P_R x, x)$ and, in Definition 2, we will use P = I, and obtain the following results from [17]:

a)
$$\operatorname{Re}(A_{c}P_{N}x,x) = \sum_{k=1}^{N} (\lambda_{k}x_{k}^{2} - g_{*}(\sum_{k=1}^{N}b_{k}x_{k})^{2}) = \underline{x}_{N}^{T}(\underbrace{\overline{A}_{N} - g_{*}\underline{b}_{N}\underline{b}_{N}^{T}}_{\overline{A}_{N}^{c}})\underline{x}_{N})$$
 where

 $\overline{A}_{N} \equiv diag[\lambda_{k}]; \underline{b}_{N} \equiv [\underline{b}_{1}..\underline{b}_{N}]^{T}; \underline{x}_{N} \equiv [x_{1}..x_{N}] \text{ where } (\overline{A}_{N}, \underline{b}_{N}, \underline{b}_{N}^{T}) \text{ is a finite dimensional system that is controllable/observable if and only if } \underline{b}_{k} \neq 0.$

- b) $(\overline{A}_N, \underline{b}_N, \underline{b}_N^T)$ is almost strictly positive real which is equivalent to $\underline{b}_N^T \underline{b}_N > 0$ and all zeros of the open-loop transfer function being stable; see e.g.[13]. We have
- c) $\underline{b}_{N}^{T} \underline{b}_{N} = \sum_{k=k}^{N} b_{k}^{2} = \|b\|^{2} = 1 > 0$ and all zeros of the open-loop are stable when

$$\overline{H}_{N} = \begin{bmatrix} \overline{A}_{N} - \lambda I & \underline{b}_{N} \\ \underline{b}_{N}^{T} & 0 \end{bmatrix} = \begin{bmatrix} \lambda_{1} - \lambda & 0 & \dots & 0 & b_{1} \\ 0 & \lambda_{2} - \lambda & \dots & 0 & b_{2} \\ \dots & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & \lambda_{N} - \lambda & b_{N} \\ b_{1} & b_{2} & \dots & b_{N} & 0 \end{bmatrix}$$

is nonsingular for all $\operatorname{Re} \lambda \ge 0$; see [14] p286.

So $(\overline{A}_{N}, \underline{b}_{N}, \underline{b}_{N}^{T})$ is ASPR if and only if

$$\det \bar{H}_{N} = (\pi_{k=1}^{N}(\lambda_{k} - \lambda)) \sum_{k=1}^{N} \frac{(-b_{k}^{2})}{\lambda_{k} - \lambda} = -\sum_{k=1}^{N} b_{k}^{2} \pi_{l=1, l \neq k}^{N}(\lambda_{k} - \lambda) \neq 0$$
(18)

for all $\operatorname{Re} \lambda \ge 0$ and $\operatorname{Re} \lambda \ne \lambda_k$ because in this application all eigenvalues are distinct and nonzero.

d) there exists
$$G_* \equiv -g_*$$
 such that $(A_c \equiv A - g_*cc^*, B \equiv b, C \equiv c^*)$ is Strictly Dissipative and
 $\operatorname{Re}(A_c x, x) \leq -\sigma ||x||^2 \forall x \in D(A)$ (19)

As long as (18) is satisfied, we can apply Theorem 3, and we have robust state tracking, $x \xrightarrow[t \to \infty]{t \to \infty} X_*$, and robust reference model tracking, $y \xrightarrow[t \to \infty]{t \to \infty} y_m$, with bounded adaptive gains $G \equiv \begin{bmatrix} G_m & G_u & G_e & G_D \end{bmatrix}$ in the presence of persistent disturbances, via the direct adaptive controller.

Example: An Unstable Heat Equation

Let
$$Ax \equiv \frac{\partial^2 x}{\partial z^2} + \beta \pi^2 x$$
 on the $D(A) \equiv \left\{ x \text{ such that } x \in \mathbb{C}^2[0,1] \text{ and } x(t,0) = x(t,1) = 0 \right\}$ which
implies that $x(t,z) = \sum_{k=1}^{\infty} e^{\lambda_k t} (x(0,z), \phi_k(z)) \phi_k(z)$ with $\lambda_k \equiv (\beta - k^2) \pi^2$ and $\phi_k \equiv \sqrt{2} \sin(k\pi x)$.

This is a heat equation with an internal source. When

$$\beta = 2 \text{ and } b = \frac{1}{\sqrt{3}} (\phi_1 + \phi_2 + \phi_3) \in S_3 \equiv sp \{\phi_1, \phi_2, \phi_3\} \subset D(A) \quad \text{this implies that}$$
$$A_N = \begin{bmatrix} \beta - 1 & 0 & 0 \\ 0 & \beta - 4 & 0 \\ 0 & 0 & \beta - 9 \end{bmatrix} \pi^2 \text{ and } b_N = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_N^T.$$

This system satisfies (6) and has $\begin{cases} poles: +\pi^2, -2\pi^2, -7\pi^2 \\ zeros: -49.35, -3.29 \end{cases}$ and so is minimum phase with $c_N b_N = 1 > 0$ and is therefore ASPR. Consequently, the direct adaptive controller in (4) will produce: output tracking

 $e_y \equiv y - y_m \xrightarrow[t \to \infty]{} 0$ with bounded adaptive gains in the presence of step disturbances.

VII. Perturbation Results

The previous results depend upon $b = P_N b \in S_N$. However, it is possible to allow $b \equiv P_N b + \varepsilon P_R b \in D(A)$; $\varepsilon \ge 0$. Define $x_N \equiv P_N x$ and $x_R \equiv P_R x$ this implies that

$$\operatorname{Re}(A(\varepsilon)_{c} x, x) = \operatorname{Re}\left(\begin{bmatrix}A_{N}^{c} & \varepsilon A_{12}\\\varepsilon A_{21} & A_{R} + \varepsilon A_{22}\end{bmatrix}\begin{bmatrix}x_{N}\\x_{R}\end{bmatrix}, \begin{bmatrix}x_{N}\\x_{R}\end{bmatrix}\right)$$
$$= \operatorname{Re}\left(\begin{bmatrix}A_{N}^{c} & 0\\0 & A_{R}\end{bmatrix}\begin{bmatrix}x_{N}\\x_{R}\end{bmatrix}, \begin{bmatrix}x_{N}\\x_{R}\end{bmatrix}\right) + \varepsilon \underbrace{\operatorname{Re}(\Delta Ax, x)}_{\leq |(\Delta Ax, x)|}$$
$$\leq -\sigma \underbrace{\left(\|x_{N}\|^{2} + \|x_{R}\|\right)^{2}}_{\|x\|^{2}} + \varepsilon \underbrace{\|\Delta A\|\|x\|^{2}}_{= -(\sigma - \varepsilon \|\Delta A\|)\|x\|^{2}}$$

And this proves:

$$\operatorname{Re}(A(\varepsilon)_{c} x, x) \leq -(\underbrace{\sigma - \varepsilon \|\Delta A\|}_{\gamma > 0}) \|x\|^{2} \text{ for all } 0 \leq \varepsilon < \frac{\sigma}{\|\Delta A\|}.$$

And we have $(A(\varepsilon)_c, B, C)$ strictly dissipative and we can apply Theorem 2 again.

Therefore, for small $\varepsilon > 0$, all previous results are still true and we do not need b entirely confined to S_N .

VIII. Conclusions

In Theorem 2 we proved a Robust Stabilization result for linear dynamic systems on infinite-dimensional Hilbert spaces under the hypothesis of almost strict dissipativity for infinite dimensional systems. This idea is an extension of the concept of m-accretivity for infinite dimensional systems; see [9] pp278-280. In Theorem 3, we showed that adaptive model tracking is possible with a very simple direct adaptive controller that knows very little specific information about the system it is controlling. This controller can also mitigate persistent disturbances. There was no use of Barbalat's lemma which requires certain signals to be uniformly continuous. However, we do not get something for nothing; we must relax the idea that all signals will converge to 0 and replace it with the idea that they will be attracted exponentially to a prescribed neighborhood whose size depends on the norm of the completely unknown disturbance. In order to cause such an infinite dimensional system to track a finite dimensional reference model, we used the idea of ideal trajectories, and in Theorem 1 we showed conditions for the existence and uniqueness of these ideal trajectories without requiring any deep knowledge of the infinite dimensional plant.

We applied these results to a general linear infinite dimensional linear systems described by self-adjoint operators with compact resolvent, in particular unstable diffusion problems using a single actuator and sensor and direct adaptive output feedback. Such systems were shown to be able to robustly track the outputs of a finite dimensional reference model in the presence of persistent disturbances.

These results do not require deep knowledge of specific properties or parameters of the system to accomplish model tracking. And they do not require that the disturbance enter through the same channels as the control. Finally, it is possible to substantially expand the results in Theorem 2 to nonlinear infinite dimensional systems, but we have elected here to take a small (baby) step forward and show the possibilities of adaptive control for infinite-dimensional systems.

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Appendix I: Proof of Theorem 2

From (16) and Pazy Cor 2.5 p107 [1], we have a well-posed system in (17) where A_c is a closed operator,

densely defined on $D(A_c) \subseteq X$ and generates a C_0 semigroup on X, and all trajectories starting in $D(A_c)$

will remain there. Hence we can differentiate signals in $D(A_C)$.

Consider the positive definite function,

$$V = \frac{1}{2} (Pe, e) + \frac{1}{2} \operatorname{tr} \left[\Delta G \gamma^{-1} \Delta G^{\mathrm{T}} \right]$$
(A.1)

where $\Delta G(t) \equiv G(t) - G^*$ and *P* satisfies (13).

Taking the time derivative of (A.1) (This can be done $\forall e \in D(A_c)$) and substituting (2a) into the result yields

$$\dot{V} = \frac{1}{2} [(PA_c e, e) + (e, PA_c e)] + (PBw, e) + tr [\Delta \dot{G} \gamma^{-1} \Delta G^{\mathrm{T}}] + (Pe, v); w \equiv \Delta Gz.$$

Invoking the equalities in the Definition 2 of Strict Dissipativity, using $x^{T}y = tr[yx^{T}]$, and substituting (17) into the last expression, we get (with $\langle e_{y}, w \rangle \equiv e_{y}^{*}w$),

$$\begin{cases} \dot{V} = \operatorname{Re}(PA_{c}e, e) + \left\langle e_{y}, w \right\rangle - a \cdot \operatorname{tr}\left[G\gamma^{-1}\Delta G^{\mathrm{T}}\right] - \underbrace{\operatorname{tr}(e_{y}z^{\mathrm{T}}\Delta G^{\mathrm{T}})}_{\left\langle e_{y}, w \right\rangle} + (Pe, v) \\ \leq -q_{\min} \left\| e \right\|^{2} - a \cdot \operatorname{tr}\left[(\Delta G + G^{*})\gamma^{-1}\Delta G^{\mathrm{T}}\right] + (Pe, v) \\ \leq -\left(q_{\min} \left\| e \right\|^{2} + a \cdot \operatorname{tr}\left[\Delta G\gamma^{-1}\Delta G^{\mathrm{T}}\right]\right) + a \cdot \left|\operatorname{tr}\left[G^{*}\gamma^{-1}\Delta G^{\mathrm{T}}\right]\right| + \left|(Pe, v)\right| \\ \leq -\left(\frac{2q_{\min}}{p_{\max}} \bullet \frac{1}{2}(Pe, e) + 2a \bullet \frac{1}{2}\operatorname{tr}\left[\Delta G\gamma^{-1}\Delta G^{\mathrm{T}}\right]\right) + a \cdot \left|\operatorname{tr}\left[G^{*}\gamma^{-1}\Delta G^{\mathrm{T}}\right]\right| + \left|(Pe, v)\right| \\ \leq -2aV + a \cdot \left|\operatorname{tr}\left[G^{*}\gamma^{-1}\Delta G^{\mathrm{T}}\right]\right| + \left|(Pe, v)\right| \end{cases}$$

Now, using the Cauchy-Schwartz Inequality $\left| \operatorname{tr} \left[G^* \gamma^{-1} \Delta G^{\mathrm{T}} \right] \right| \leq \left\| G^* \right\|_2 \left\| \Delta G \right\|_2$ and

$$|(Pe,v)| \le \left\| P^{\frac{1}{2}} v \right\| \left\| P^{\frac{1}{2}} e \right\| = \sqrt{(Pv,v)} \bullet \sqrt{(Pe,e)}$$

We have

$$\dot{V} + 2aV \le a \cdot \left\|G^*\right\|_2 \left\|\Delta G\right\|_2 + \sqrt{p_{\max}} \left\|v\right\| \sqrt{(Pe, e)}$$

$$\le a \cdot \left\|G^*\right\|_2 \left\|\Delta G\right\|_2 + (\sqrt{p_{\max}}M_v) \sqrt{(Pe, e)}$$

$$\le (a \left\|G^*\right\|_2 + \sqrt{p_{\max}}M_v) \sqrt{2} \underbrace{\left[\frac{1}{2}(Pe, e) + \frac{1}{2} \left\|\Delta G\right\|_2^2\right]^{\frac{1}{2}}}_{V^{\frac{1}{2}}}$$

Therefore,

$$\frac{\dot{V} + 2aV}{V^{\frac{1}{2}}} \le (a \left\| G^* \right\|_2 + \sqrt{p_{\max}} M_v) \sqrt{2}$$

Now, using the identity $\operatorname{tr}[ABC] = \operatorname{tr}[CAB]$,

$$\begin{split} \left\| G^* \right\|_2 &\equiv \left[\operatorname{tr} \left(G^* \gamma^{-1} (G^*)^T \right) \right]^{\frac{1}{2}} = \left[\operatorname{tr} \left((G^*)^T G^* \gamma^{-1} \right) \right]^{\frac{1}{2}} \\ &\leq \left[\left(\operatorname{tr} \left((G^*)^T G^* (G^*)^T G^* \right) \right)^{\frac{1}{2}} \left(\operatorname{tr} (\gamma^{-1} \gamma^{-1}) \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \\ &= \left[\operatorname{tr} \left(G^* (G^*)^T \right) \right]^{\frac{1}{2}} \left[\operatorname{tr} (\gamma^{-1}) \right]^{\frac{1}{2}} \\ &\leq \frac{M_{\nu}}{aM_G} \bullet M_G = \frac{M_{\nu}}{a} \end{split}$$

which implies that

$$\frac{\dot{V} + 2aV}{V^{\frac{1}{2}}} \le \left(1 + \sqrt{p_{\max}}\right) M_{\nu} \sqrt{2}$$
 (A.2)

From

$$\frac{d}{dt}(2e^{at}V^{\frac{1}{2}}) = e^{at}\frac{\dot{V}+2aV}{V^{\frac{1}{2}}}$$
$$\leq e^{at}\left(1+\sqrt{p_{\max}}\right)M_{\nu}\sqrt{2}$$

Integrating this expression we have:

$$e^{at}V(t)^{1/2} - V(0)^{1/2} \le \frac{\left(1 + \sqrt{p_{\max}}\right)M_{\nu}}{a} \left(e^{at} - 1\right).$$

Therefore,

$$V(t)^{1/2} \le V(0)^{1/2} e^{-at} + \frac{\left(1 + \sqrt{p_{\max}}\right) M_{\nu}}{a} \left(1 - e^{-at}\right)$$
(A.3)

The function V(t) is a norm function of the state e(t) and matrix G(t). So, since $V(t)^{1/2}$ is bounded for all t, then e(t) and G(t) are bounded. We also obtain the following inequality:

$$\sqrt{p_{\min}} \| e(t) \| \leq V(t)^{1/2}$$

Substitution of this into (A.3) gives us an exponential bound on state $e(\tau)$:

$$\|e(t)\| \le \frac{e^{-at}}{\sqrt{p_{\min}}} V(0)^{1/2} + \frac{\left(1 + \sqrt{p_{\max}}\right)M_{\nu}}{a\sqrt{p_{\min}}} \left(1 - e^{-at}\right)$$
(A.4)

Taking the limit superior of (A.4), we have

$$\lim_{\tau \to \infty} \left\| e(t) \right\| \le \frac{\left(1 + \sqrt{p_{\max}} \right)}{a \sqrt{p_{\min}}} M_{\nu} \equiv R_*.$$
(A.5)

And the proof is complete.