Distributed Parameter Optimal Control by Adjoint Aeroelastic Differential Operators for Mode Suppression Control

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This paper presents a distributed parameter optimal control method. The state-space equation of wing bending-torsion aeroelasticity is formulated using aeroelastic differential operators. The optimal control problem is formulated as linear quadratic regulator in a Hilbert inner product space. The solution to the optimal control problem leads to a differential operator Ricatti equation. Using a finite-series modal approximation, the differential operator Ricatti equation is converted into a spatially dependent Ricatti equation. The solution of this spatially dependent Ricatti equation is used to compute the distributed parameter optimal control which is spatially continuous. A flutter suppression control application is demonstrated on a flexible wing transport aircraft outfitted with a variable camber continuous trailing edge flap system.

I. Introduction

Aeroelasticity deals with an important flight phenomenon in which interactions between vehicle aerodynamics and structures become strongly coupled. Flutter mechanisms are manifestation of aeroelasticity. Modern aircraft are increasingly designed to take advantage of lightweight materials to improve aircraft aerodynamic performance. One major concern with the use of light weight materials for load carrying members in a wing structure is the increased flexibility. This flexibility can affect the flutter margin in a traditional sense but also can adversely impact aircraft aerodynamic performance. Thus, the use of lightweight materials requires aircraft designers to pay a much closer attention in modern aircraft to the issue of aeroelasticity which not only can affect aircraft safety but also aircraft aerodynamic performance. As a result, multi-disciplinary design, analysis, and optimization (MDAO) plays an important role in modern aircraft design. Flight control is an important consideration in an MDAO approach particularly as the flexibility becomes a dominant factor in aircraft design. Aeroelastic stability augmentation and gust load alleviation control may be needed to improve aircraft reliability. For example, the new Boeing 787 aircraft has a wing flexibility twice as much as that of an older-generation transport aircraft such as the Boeing 757. As a result, the flight control system of the Boeing 787 aircraft includes a gust load alleviation control function to improve the ride quality.

Flutter suppression control has been well studied. Many approaches based on classical control, robust control, and adaptive control have been developed. Typically, an aeroelastic system is considered to be an infinite dimensional system. What this means is that the system has infinite number of natural modes, each of which is associated with an eigenvalue. In contrast, rigid-body aircraft flight dynamics are considered as finite dimensional systems which only possess finite number of natural modes such as the short-period mode. Mathematically, infinite dimensional systems are governed by partial differential equations, whereas finite dimensional systems are described by ordinary differential equations.

In engineering analysis, infinite dimensional systems are usually approximated by finite dimensional systems by discretization. The order of the approximate finite dimensional systems must be sufficiently high in order to improve

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the approximation. As the order of a finite dimensional system increases, control of the system becomes more complex. Furthermore, only a few states of a finite dimensional system that approximates an infinite dimensional system may be available for feedback control. As a result, a well-known technique employed in control of infinite dimensional systems is model reduction. Once an infinite dimensional system is approximated as a high-order finite dimensional system, the finite dimensional system is further approximated by a lower-order system called a reduced-order model. Balanced truncation is a common technique for model reduction. The reduced-order model is then used to develop a control system for the original infinite dimensional system.

While the model reduction approach is well understood, the possibility of introducing model reduction errors in a control design as a consequence of the elimination of extra state variables from the original high-order finite dimensional system may exist. Therefore, the control design must be ensured high degree of robustness in safeguarding the potential introduction of improper control actions into an aeroelastic system as a result of the model reduction errors. An equivalent philosophy in control design of infinite dimensional systems may be considered. A control design can be developed for an infinite dimensional system without approximating the system using a finite dimensional representation. Once a control design has been developed, the infinite dimensional system which includes the control design is discretized into a finite-dimensional system. This approach is referred to as a distributed parameter control approach whereby a control design is dependent on a spatially continuous or distributed state variables. In actual implementation, the distributed state variables would have to be approximated by a finite dimensional system which includes an observer design to estimate all unmeasured state variables. The distributed parameter control represents a viable approach to the standard reduced-order model control for infinite dimensional systems.

In this paper, we will present an aeroelastic system for wing bending and torsion as an infinite dimensional system. An aeroelastic differential operator is introduced that enables the partial differential equations which govern the aeroelastic system to be cast in a state space form. The eigenvalues of the differential operator determine stability of the aeroelastic system. An optimal control approach is then developed for this aeroelastic system based on the aeroelastic differential operator.

II. Wing Aeroelasticity and Aeroelastic Differential Operator

To motivate the idea, we first introduce an aeroelastic problem of wing torsion. The aeroelastic angle of attack includes the rigid-body angle of attack, induced angle of attack due to local downwash, and the wing twist. Neglecting the induced angle of attack, the aeroelastic angle of attack at the aerodynamic center of a wing section is given by

$$\alpha_{ac} = \frac{\alpha}{\cos \Lambda} - \gamma + \frac{x_{ac} q}{V_\infty \cos \Lambda} - \Theta + \frac{e \Theta_t}{V_\infty \cos \Lambda}$$

(1)

where $\alpha$ is the aircraft angle of attack, $\gamma$ is the wing pre-wist angle, $q$ is aircraft pitch rate, $x_{ac}$ is the forward distance of the aircraft center of gravity from the aerodynamic center of a wing section, $e$ is the forward distance of the aerodynamic center from the elastic center, $\Lambda$ is the wing swept angle, $V_\infty$ is the airspeed.

The aeroelastic angle of attack at the mid-chord location is given by

$$\alpha_{mc} = \frac{\alpha}{\cos \Lambda} - \gamma + \frac{x_m q}{V_\infty \cos \Lambda} - \Theta - \frac{e_m \Theta_t}{V_\infty \cos \Lambda}$$

(2)

where $x_m$ is the forward distance of the aircraft center of gravity from the mid-chord location of a wing section and $e_m$ is the forward distance of the aeroelastic center from the mid-chord location.

Based on the classical aeroelasticity theory of unsteady aerodynamics, the wing section lift coefficient is given by

$$c_{L_{ac}} = c_{L_{ac}} C(k) \alpha_{ac} + c_{L_d} \delta$$

(3)

where $k = \frac{\omega c}{V_\infty}$ is the reduced frequency parameter, $\omega$ is the frequency of wing oscillations, $c$ is the section chord, $c_{L_{ac}}$ is the section lift curve slope, $c_{L_d}$ is the section lift derivative, and $\delta$ is the control surface deflection.

For simplification we will neglect the actuator dynamics in the analysis.

The function $C(k)$ is the Theodorsen’s complex-valued function which is also expressed in terms of Bessel functions as

$$C(k) = F(k) + iG(k)$$

(4)

where $F(k) > 0$ and $G(k) < 0$.

When $k = 0$, the airfoil motion is steady and $C(k)$ is real and unity. As $k$ increases, there is a phase lag introduced as the magnitude of $G(k)$ increases as shown in Fig. 1. The limiting values of $F(k)$ and $G(k)$ are $1/2$ and $0$ as $k \to \infty$.2
Consider the case when the wing undergoes a simple harmonic motion
\[ \alpha_{ac} = \bar{\alpha} e^{i\omega t} \]  
(5)

Then
\[ \dot{\alpha}_{ac} = i\omega \bar{\alpha} e^{i\omega t} = i\omega \alpha_{ac} \]  
(6)

Thus the wing section lift coefficient due to unsteady aerodynamics is expressed as
\[ c_L_{ac} = c_L \alpha_{ac} F(k) + c_{L\alpha} \frac{\alpha_{ac} G(k)}{k} + c_L \dot{\delta} \]  
(7)

In addition, the apparent mass of the air contributes to the lift force acting at the mid-chord location as follows:
\[ c_{L_{mc}} = \frac{\pi \dot{\alpha}_{mc} c}{2V_\infty} \]  
(8)

The total section lift coefficient is
\[ c_L = c_{L_{ac}} + c_{L_{mc}} \]  
(9)

The section pitching moment coefficient is evaluated as
\[ c_m = c_{m_{ac}} + \frac{e}{c} c_{L_{ac}} - \frac{m}{c} c_{L_{mc}} + c_{m\delta} \dot{\delta} \]  
(10)

where \( c_{m_{ac}} \) is the section pitching moment coefficient at the aerodynamic center and \( c_{m\delta} \) is the section pitching moment derivative at the aerodynamic center due to the control surface deflection \( \delta \).

The aeroelastic equation for wing torsion is given by
\[ (GJ\Theta_t)_y = c_{m_{ac}} q_\infty \cos^2 \Lambda c^2 - mge_{cg} + mr_k^2 \Theta_{tt} \]  
(11)

where \( \Theta \) is the wing twist, \( GJ \) is the torsional rigidity, \( q_\infty \) is the dynamic pressure, \( m \) is the wing mass per unit length, \( e_{cg} \) is the forward distance of the elastic center from the center of mass, \( r_k \) is the section radius of gyration, and the subscripts \( y \) and \( t \) denote the partial derivatives with respect to \( y \), the coordinate along the wing span, and time \( t \).

Substituting in the pitching moment coefficient results in
\[ (GJ\Theta_t)_y = \left[ c_{m_{ac}} + c_{L_{ac}} \left( \frac{\alpha}{\cos \Lambda} - \gamma + \frac{x a q}{V_\infty \cos \Lambda} - \Theta + \frac{e \Theta_t}{V_\infty \cos \Lambda} \right) \right] F(k) \]
\[ + c_{L_{ac}} \left( \frac{\alpha}{\cos \Lambda} + \frac{x a q}{V_\infty \cos \Lambda} - \Theta_t + \frac{e \Theta_t}{V_\infty \cos \Lambda} \right) \frac{c}{2V_\infty} \frac{G(k)}{k} + c_{L\dot{\delta}} \dot{\delta} \]
\[ - \frac{m}{2V_\infty} \pi c \left( \frac{\alpha}{\cos \Lambda} + \frac{x a q}{V_\infty \cos \Lambda} - \Theta_t + \frac{e \Theta_t}{V_\infty \cos \Lambda} \right) \left[ q_\infty \cos^2 \Lambda c - mge_{cg} + mr_k^2 \Theta_{tt} \right] \]  
(12)
Consider the following cases:

A. Quasi-Steady Aerodynamics

The quasi-steady aerodynamic assumption is invoked whereby $F(k) \approx 1$ and $G(k) \approx 0$. Then

$$
(GJ\Theta)_y = \left[cc_{mac} + ecL_{at}\left(\frac{\alpha}{\cos \Lambda} - \gamma + \frac{x_a q}{V_m \cos \Lambda} - \Theta + \frac{e \Theta_j}{V_m \cos \Lambda}\right) \right.

- e_m \frac{\pi c}{2V_m} \left(\frac{\alpha}{\cos \Lambda} + \frac{x_m q}{V_m \cos \Lambda} - \Theta_j - \frac{e_m \Theta_j}{V_m \cos \Lambda}\right) + \left(cc_{mg} + ecL_{gb}\right) \delta \left] q_m \cos^2 \Lambda - mge_c + mr^2 \Theta_t \right.

\right)
$$

Without considering rigid-body longitudinal dynamic coupling, this partial differential equation can be recast as

$$
\frac{\partial}{\partial t} \left[ \begin{array}{c} \Theta \\ \frac{\partial \Theta}{\partial t} \end{array} \right] = \left[ \begin{array}{c} 0 \\ \frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \left(GJ \frac{\partial}{\partial t} \right) + ecL_{at} q_m \cos^2 \Lambda \right) \left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right) \frac{1}{\left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right)} \left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right) \frac{q_m \cos^2 \Lambda}{\left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right)} \right]

$$

where $m_a = \frac{\pi be^2}{4}$ is the running mass of the air cylindrical volume surrounding the wing, and,

$$
f(y,t) = \left[ -cc_{mac} - ecL_{at} \left(\frac{\alpha}{\cos \Lambda} - \gamma + \frac{x_a q}{V_m \cos \Lambda} \right) + \frac{\pi ec}{2V_m \cos \Lambda} \left(\frac{x_m q}{V_m \cos \Lambda} - \Theta_j - \frac{x_m q}{V_m \cos \Lambda}\right) q_m \cos^2 \Lambda + mge_c \right]
$$

Let $x(y,t) = \left[ \begin{array}{c} \Theta \\ \frac{\partial \Theta}{\partial t} \end{array} \right]$ be a distributed state vector and $u = \delta(y,t)$ be a distributed control variable, then the aeroelastic equation can be written as

$$
\frac{\partial x}{\partial t} = Ax + Bu + w
$$

where

$$
A = \left[ \begin{array}{c} \frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \left(GJ \frac{\partial}{\partial t} \right) + ecL_{at} q_m \cos^2 \Lambda \right) \left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right) \frac{1}{\left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right)} \left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right) \frac{q_m \cos^2 \Lambda}{\left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right)} \right]

$$

$$
B = \left[ \begin{array}{c} 0 \\ f(y,t) \end{array} \right]
$$

$$
w = \left[ \begin{array}{c} 0 \\ f(y,t) \end{array} \right]
$$

The matrix $A$ is a linear operator that operates on $x$ and is dependent on the partial derivatives with respect to $y$. Therefore, it represents an aeroelastic differential operator. Stability of an aeroelastic system can be determined by the eigenvalues of the aeroelastic differential operator $A$.

The resolvent set is a space of possible solutions that solve the eigenvalue problem of the aeroelastic partial differential equation defined by $(\lambda I - A)x = 0$. This eigenvalue problem is formulated as

$$
(\lambda I - A)x = \left[ \begin{array}{c} \lambda \\ -\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \left(GJ \frac{\partial}{\partial t} \right) + ecL_{at} q_m \cos^2 \Lambda \right) \left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right) \frac{1}{\left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right)} \left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right) \frac{q_m \cos^2 \Lambda}{\left(\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right) \right)} \right]

$$

where $\lambda$ is the eigenvalue.
Expanding this equation yields

\[- \frac{1}{mr_k^2 + m_a e_m^2 \cos \Lambda} \frac{\partial}{\partial y} \left( GJ \frac{\partial \Theta}{\partial y} \right) - \frac{e c_{La} q_\infty \cos^2 \Lambda e \Theta}{mr_k^2 + m_a e_m^2 \cos \Lambda} + \left[ \lambda + \left( \frac{e^2 c_{La}}{\cos \Lambda} + \frac{\pi e_m c}{2} \right) \frac{q_\infty \cos^2 \Lambda c}{(mr_k^2 + m_a e_m^2 \cos \Lambda) V_\infty} \right] \frac{\partial \Theta}{\partial t} = 0 \]  

(21)

Thus, Eq. (117) becomes the eigenvalue differential equation for the aeroelastic system

\[ \Theta (y, t) = a \phi (y) e^{\lambda t} \]  

(23)

subject to the boundary conditions \( \phi (0) = 0 \) and \( \phi' (L) = 0 \).

By letting

\[ p^2 = \frac{e c_{La} q_\infty \cos^2 \Lambda c}{GJ} - \left( \frac{mr_k^2 + m_a e_m^2 \cos \Lambda} \right) \lambda + \left( \frac{e^2 c_{La}}{\cos \Lambda} + \frac{\pi e_m c}{2} \right) \frac{q_\infty \cos^2 \Lambda c}{V_\infty} \]  

(25)

Equation (24) is recognized as a Sturm-Liouville equation

\[ - \left[ GJ \phi' (y) \right]' = p^2 GJ \phi (y) \]  

(26)

Therefore, the eigenvalue \( p \) is positive and the eigenfunction \( \phi (y) \) is orthogonal.

In general, the eigenvalues can be determined numerically using methods such as the Galerkin’s method and finite-element method. For the purpose of illustration, we consider a simple case when all parameters are constant. Then the Sturm-Liouville equation has a closed-form solution.

Equation (24) is of the form

\[ \phi'' (y) + p^2 \phi (y) = 0 \]  

(27)

where

\[ p^2 = \frac{e c_{La} q_\infty \cos^2 \Lambda c}{GJ} - \left( \frac{mr_k^2 + m_a e_m^2 \cos \Lambda} \right) \lambda + \left( \frac{e^2 c_{La}}{\cos \Lambda} + \frac{\pi e_m c}{2} \right) \frac{q_\infty \cos^2 \Lambda c}{V_\infty} \]  

(28)

The general solution with the specified boundary conditions is given by the following eigenfunction

\[ \phi (y) = \sin p y \]  

(29)

where the eigenvalues \( p \) are the roots of \( \cos p L = 0 \) which yield \( p_n = \frac{(n+\frac{1}{2}) \pi}{L} \) for \( n = 0, 1, \ldots, \infty \).

The eigenvalues \( \lambda \) are then computed from the following characteristic equation

\[ \lambda^2 + \left( \frac{e^2 c_{La}}{\cos \Lambda} + \frac{\pi e_m c}{2} \right) \frac{q_\infty \cos^2 \Lambda c}{(mr_k^2 + m_a e_m^2 \cos \Lambda) V_\infty} \lambda - \frac{e c_{La} q_\infty \cos^2 \Lambda c}{(mr_k^2 + m_a e_m^2 \cos \Lambda) V_\infty} + \omega_n^2 = 0 \]  

(30)

where \( \omega_n^2 = \frac{GJ (n+\frac{1}{2})^2}{(mr_k^2 + m_a e_m^2 \cos \Lambda) L^2} \) is the natural frequency of a structural dynamic mode.

The solution of the characteristic equation yields

\[ \lambda_n = - \left( \frac{e^2 c_{La}}{\cos \Lambda} + \frac{\pi e_m c}{2} \right) \frac{q_\infty \cos^2 \Lambda c}{2 (mr_k^2 + m_a e_m^2 \cos \Lambda) V_\infty} \]

\[ \pm \sqrt{ \left[ \frac{e^2 c_{La}}{\cos \Lambda} + \frac{\pi e_m c}{2} \right] \frac{q_\infty \cos^2 \Lambda c}{2 (mr_k^2 + m_a e_m^2 \cos \Lambda) V_\infty} + \frac{e c_{La} q_\infty \cos^2 \Lambda c}{(mr_k^2 + m_a e_m^2 \cos \Lambda) V_\infty} - \omega_n^2 } \]  

(31)
The following observation can be made. If the following condition is satisfied
\[ ec_L q_\infty \cos^2 \Lambda c > \frac{GJ (n + \frac{1}{2})^2 \pi^2}{L^2} \]  
(32)
then \( \Re(\lambda_n) > 0 \).

The aeroelastic system is then unstable. The lowest critical airspeed at which the system is neutrally stable is known as the torsional divergence speed which corresponds to the torsional divergence dynamic pressure
\[ q_d = \frac{\pi^2 GJ}{4L^2 ec_L \cos^2 \Lambda c} \]  
(33)
It is observed that aeroelastic instability is usually associated with low-order modes for which \( n \) is small. As \( n \to \infty \), then
\[ \lim_{n \to \infty} \lambda_n = - \left( \frac{\pi^2 c_L a_t}{\cos \Lambda} + \frac{\pi \epsilon m c}{2} \right) \frac{q_\infty \cos^2 \Lambda c}{2 (m r_{\alpha}^2 + m_a e_{\alpha}^2 \cos \Lambda) V_\infty} \pm i \omega_{\epsilon t} \]  
(34)
The aeroelastic system is then stable.
The general solution for the aeroelastic partial differential equation is
\[ \Theta(y, t) = \sum_{n=0}^{N \to \infty} a_n \phi_n(y) e^{i \omega_{\epsilon t}} = \sum_{n=0}^{N \to \infty} \phi_n(y) \theta_n(t) \]  
(35)
where \( \phi_n(y) \) is a modal shape function and \( \theta_n(t) \) is a generalized coordinates.

The modal shape functions \( \theta_n(y) \) forms a set of orthonormal basis functions that span the entire space \( \mathcal{S} \). If the closure of the range of the operator \( (A - \lambda I) \) is the whole space \( \mathcal{S} \) and the inverse operator \( (A - \lambda I)^{-1} \) is a bounded operator, then \( \lambda(A) \) is said to belong to the resolvent set of the operator \( A \). Then, it follows that the Laplace transform inverse operator \( (sI - A)^{-1} \) is also bounded. The open-loop transfer function operator of the state-space operator equation is expressed as
\[ \frac{u}{x} = (sI - A)^{-1} B \]  
(36)
If \( \lambda(A) \) is in the resolvent set of the operator \( A \), then it follows that the transfer function operator is a bounded operator. This implies that if \( \|u(y, t)\| \in \mathcal{L}_2 \), the space of Lebesgue square-integrable functions, is bounded, then \( \|x(y, t)\| \in \mathcal{L}_2 \) is also bounded. The Hilbert inner-product space is one such space of Lebesgue square-integrable functions where
\[ \|x\| = (x, x) = \int_0^L x^\top x dy \]  
(37)
Thus, the aeroelastic differential operator \( A \) has the same interpretation as the transition matrix in the standard state-space formulation. The transfer function operator \( (sI - A)^{-1} \) has the same meaning as that in the final dimensional systems. The aeroelastic differential operator \( A \) possesses infinite number of eigenvalues since the aeroelastic system is infinite dimensional.

B. Low-Frequency Unsteady Aerodynamics

At low frequencies of wing vibration, the Theodorsen’s function may be approximated as
\[ C(k) \approx (1 + F_0k) + iG_0k \]  
(38)
where \( F_0 = dF/dk \approx -1.68 \) and \( G_0 = dG/dk \approx -1.72 \) evaluated at \( k = 0 \) for \( k \leq 0.1 \).

Then
\[ \langle GJ \Theta_t \rangle_y = \left[ cc_{mc} + ec_L a_t \left( \frac{\alpha}{\cos \Lambda} - \gamma + \frac{x_{\alpha} q}{V_\infty \cos \Lambda} - \Theta + \frac{e \Theta_t}{V_\infty \cos \Lambda} \right) (1 + F_0k) \right. \]
\[ + ec_L a_t \left( \frac{\alpha}{\cos \Lambda} + \frac{x_{\alpha} q}{V_\infty \cos \Lambda} - \Theta_t + \frac{e \Theta_{tt}}{V_\infty \cos \Lambda} \right) \frac{G_0}{2V_\infty} + ec_L \delta \]
\[ - \frac{\pi e}{2V_\infty} \left( \frac{\alpha}{\cos \Lambda} + \frac{x_{\alpha} q}{V_\infty \cos \Lambda} - \Theta_t - \frac{e m \Theta_{tt}}{V_\infty \cos \Lambda} \right) + cc_m \delta \]  
\[ \left. q_\infty \cos^2 \Lambda c - mg_{\epsilon x} + mr_{\alpha}^2 \Theta_{tt} \right] \]  
(39)
where \( r_a^2 = (\frac{\varepsilon c_{Ld} G_0}{\pi} + e_n^2) \cos \Lambda \) is the radius of gyration of the cylindrical air volume surrounding the wing.

The eigenvalue differential equation for the aeroelastic system is governed by

\[
- \left[ GI \phi'(y) \right] - e c_{Ld} q_\infty \cos^2 \Lambda \phi(y) (1 + F_0k) \\
+ \left\{ \left( m r_k^2 + m_a r_a^2 \right) \Lambda + \left[ \frac{e^2 c_{Ld} (1 + F_0k)}{2 V_\infty \cos \Lambda} - \frac{e c_{Ld} G_0}{2} + \frac{\pi e_m c}{2} \right] q_\infty \cos^2 \Lambda \right\} \Lambda \phi(y) = 0
\] (41)

subject to the boundary conditions \( \phi(0) = 0 \) and \( \phi'(L) = 0 \).

For illustration, we assume all parameters are constant. Let \( \Lambda = \sigma + i \omega \), then the characteristic equation is obtained as

\[
\left\{ (\sigma + i \omega) + \left[ \frac{e^2 c_{Ld} (2 V_\infty + F_0 \omega c)}{2 V_\infty \cos \Lambda} - \frac{e c_{Ld} G_0}{2} + \frac{\pi e_m c}{2} \right] \frac{q_\infty \cos^2 \Lambda}{(m r_k^2 + m_a r_a^2) V_\infty} \right\} (\sigma + i \omega) \\
- \frac{e c_{Ld} q_\infty \cos^2 \Lambda (2 V_\infty + F_0 \omega c)}{2 (m r_k^2 + m_a r_a^2) V_\infty} + \frac{G I (n + \frac{1}{2})^2 \pi^2}{(m r_k^2 + m_a r_a^2) L^2} = 0
\] (42)

The real part and imaginary part of the characteristic equation yield the following equations

\[
\sigma^2 - \omega^2 + \sigma \left[ \frac{e^2 c_{Ld} (2 V_\infty + F_0 \omega c)}{2 V_\infty \cos \Lambda} - \frac{e c_{Ld} G_0}{2} + \frac{\pi e_m c}{2} \right] \frac{q_\infty \cos^2 \Lambda}{(m r_k^2 + m_a r_a^2) V_\infty} \\
- \frac{e c_{Ld} q_\infty \cos^2 \Lambda (2 V_\infty + F_0 \omega c)}{2 (m r_k^2 + m_a r_a^2) V_\infty} + \frac{G I (n + \frac{1}{2})^2 \pi^2}{(m r_k^2 + m_a r_a^2) L^2} = 0
\] (43)

\[
2 \sigma + \left[ \frac{e^2 c_{Ld} (2 V_\infty + F_0 \omega c)}{2 V_\infty \cos \Lambda} - \frac{e c_{Ld} G_0}{2} + \frac{\pi e_m c}{2} \right] \frac{q_\infty \cos^2 \Lambda}{(m r_k^2 + m_a r_a^2) V_\infty} = 0
\] (44)

Solving for \( \sigma \) and \( \omega \) yields

\[
\sigma = - \left[ \frac{e^2 c_{Ld} (2 V_\infty + F_0 \omega c)}{2 V_\infty \cos \Lambda} - \frac{e c_{Ld} G_0}{2} + \frac{\pi e_m c}{2} \right] \frac{q_\infty \cos^2 \Lambda}{2 (m r_k^2 + m_a r_a^2) V_\infty}
\] (45)

\[
\omega = - \frac{a_1}{2 a_2} + \sqrt{\left( \frac{a_1}{2 a_2} \right)^2 + \frac{\omega_n^2 - a_0}{a_2}}
\] (46)

where \( \omega_n^2 = \frac{G I (n + \frac{1}{2})^2 \pi^2}{(m r_k^2 + m_a r_a^2) L^2} \) and

\[
a_0 = V_\infty \left( \frac{1}{e} - \frac{G_0 \cos \Lambda}{2e} + \frac{\pi e_m \cos \Lambda}{2e e^2 c_{Ld}} \right)^2 \left[ \frac{m_a e^2 c_{Ld} \cos \Lambda}{\pi (m r_k^2 + m_a r_a^2)} \right]^2 + \frac{e c_{Ld} q_\infty \cos^2 \Lambda}{m r_k^2 + m_a r_a^2}
\] (47)

\[
a_1 = V_\infty F_0 \left( \frac{1}{e} - \frac{G_0 \cos \Lambda}{2e} + \frac{\pi e_m \cos \Lambda}{2e e^2 c_{Ld}} \right) \left[ \frac{m_a e^2 c_{Ld} \cos \Lambda}{\pi (m r_k^2 + m_a r_a^2)} \right]^2 + \frac{m_a e c_{Ld} \cos^2 \Lambda}{\pi (m r_k^2 + m_a r_a^2)}
\] (48)

\[
a_2 = 1 + \left( \frac{2}{4 \pi^2} \right) \left[ \frac{m_a e^2 c_{Ld} F_0 \cos \Lambda}{m r_k^2 + m_a r_a^2} \right]^2
\] (49)

Note that we recover the eigenvalue solution for the aeroelastic problem of wing torsion with quasi-steady aerodynamics by setting \( F_0 = 0 \) and \( G_0 = 0 \) in the expressions above.
Consider a more general aeroelastic problem of combined wing bending and torsion. The aeroelastic angles of attack are now modified to include the effect of bending as

\[
\alpha_{ac} = \frac{\alpha}{\cos \Lambda} - \gamma + \frac{x_a q}{V_\infty \cos \Lambda} - \Theta + \frac{e \Theta_t}{V_\infty \cos \Lambda} - W_t \tan \Lambda - \frac{W_t}{V_\infty \cos \Lambda} \tag{50}
\]

\[
\alpha_{mt} = \frac{\alpha}{\cos \Lambda} - \gamma + \frac{x_m q}{V_\infty \cos \Lambda} - \Theta - e_m \Theta_t - W_t \tan \Lambda - \frac{W_t}{V_\infty \cos \Lambda} \tag{51}
\]

where \( W \) is the wing vertical bending deflection.

The combined bending and torsion aeroelastic equations are given by

\[
(GJ \Theta_t) = \left[ c_c m_c + c e \right] \left( \frac{\alpha}{\cos \Lambda} - \gamma + \frac{x_a q}{V_\infty \cos \Lambda} - \Theta + \frac{e \Theta_t}{V_\infty \cos \Lambda} - W_t \tan \Lambda - \frac{W_t}{V_\infty \cos \Lambda} \right) F(k)
\]

\[
+ e c \left( \frac{\alpha}{\cos \Lambda} - \Theta + \frac{e \Theta_t}{V_\infty \cos \Lambda} - W_t \tan \Lambda - \frac{W_t}{V_\infty \cos \Lambda} \right) \frac{c}{2 V_\infty} G(k) + (c c_m + c e \delta) \delta
\]

\[
- e_m \frac{\pi c}{2 V_\infty} \left( \frac{\alpha}{\cos \Lambda} - \Theta - e_m \Theta_t - W_t \tan \Lambda - \frac{W_t}{V_\infty \cos \Lambda} \right) q \cos^2 \Lambda c
\]

\[
- m c g_e + m c \delta_\delta \delta - m c g_e W_t
\]

\[
(EI W_{yy}) = \left[ c_L a \left( \frac{\alpha}{\cos \Lambda} - \gamma + \frac{x_a q}{V_\infty \cos \Lambda} - \Theta + \frac{e \Theta_t}{V_\infty \cos \Lambda} - W_t \tan \Lambda - \frac{W_t}{V_\infty \cos \Lambda} \right) F(k)
\]

\[
+ c_L a \left( \frac{\alpha}{\cos \Lambda} - \Theta + \frac{e \Theta_t}{V_\infty \cos \Lambda} - W_t \tan \Lambda - \frac{W_t}{V_\infty \cos \Lambda} \right) \frac{c}{2 V_\infty} G(k) + c_L \delta
\]

\[
+ \frac{\pi c}{2 V_\infty} \left( \frac{\alpha}{\cos \Lambda} - \Theta - e_m \Theta_t - W_t \tan \Lambda - \frac{W_t}{V_\infty \cos \Lambda} \right) q \cos^2 \Lambda c - m g - m W_t + m c g_e \Theta_t
\]

Let \( x(y,t) = \left[ \Theta \ W \ \frac{\partial \Theta}{\partial t} \ \frac{\partial w}{\partial t} \right]^T \) be the distributed state vector, then the aeroelastic differential operator \( A \), the control sensitivity matrix \( B \), and the aerodynamic forcing function \( w \) are given by

\[
A(k) = \begin{bmatrix}
0 & I \\
-M^{-1} K & -M^{-1} C
\end{bmatrix}
\tag{54}
\]

\[
B(k) = \begin{bmatrix}
0 \\
-M^{-1} F_\delta
\end{bmatrix}
\tag{55}
\]

\[
w(k) = \begin{bmatrix}
0 \\
-M^{-1} F
\end{bmatrix}
\tag{56}
\]

where \( r_a^2 = \left( e c_L a \frac{G(k)}{k} + e_m^2 \right) \cos \Lambda; \ e_a = \left( e c_L a \frac{G(k)}{k} - e_m \right) \cos \Lambda \); and \( M, C, K, F_\delta \), and \( F \) are the mass matrix, damping
operator, stiffness operator, control force sensitivity vector, and force vector, respectively, and are defined as

$$
M(k) = \begin{bmatrix}
    m_k^2 + m_a r_a^2 & -m_e g - m_a e_a \\
    -m_e g - m_a e_a & m + m_a \left( \frac{\epsilon_a}{k} \frac{G(k)}{k} + 1 \right) \cos \Lambda
\end{bmatrix}
$$

(57)

$$
C(k) = \begin{bmatrix}
    \left( \frac{\epsilon_a}{k} \frac{G(k)}{k} \right) \frac{q_0 \cos^2 \Lambda}{V_c} & \left( \frac{\epsilon_a}{k} \frac{G(k)}{k} \right) \frac{q_0 \cos^2 \Lambda}{V_c} \\
    \left( \frac{\epsilon_a}{k} \frac{G(k)}{k} \right) \frac{q_0 \cos^2 \Lambda}{V_c} & \left( \frac{\epsilon_a}{k} \frac{G(k)}{k} \right) \frac{q_0 \cos^2 \Lambda}{V_c}
\end{bmatrix}
$$

(58)

$$
K(k) = \begin{bmatrix}
    -\frac{\partial}{\partial y} (GJ \frac{\partial}{\partial y}) - ec_L F(k) q_0 \cos^2 \Lambda & -ec_L \tan \Lambda F(k) \frac{q_0 \cos^2 \Lambda}{\partial y} \\
    ec_L F(k) q_0 \cos^2 \Lambda & \frac{\partial}{\partial y} \left( EI \frac{\partial^2}{\partial y^2} \right) + ec_L \tan \Lambda F(k) \frac{q_0 \cos^2 \Lambda}{\partial y}
\end{bmatrix}
$$

(59)

$$
F_\delta = \begin{bmatrix}
    - (cc_m + ec_L a) \frac{q_0 \cos^2 \Lambda}{c_L q_0 \cos^2 \Lambda} \\
    \frac{c_L a}{\cos \Lambda} + \frac{\epsilon_a}{k} \frac{q_0 \cos^2 \Lambda}{V_c \cos \Lambda}
\end{bmatrix}
$$

(60)

$$
F(k) = \begin{bmatrix}
    \left[ cc_m + ec_L \left( \frac{\alpha}{\cos \Lambda} + \gamma + \frac{\epsilon_a}{V_c \cos \Lambda} \right) \right] F(k) + \left( ec_L \left( \frac{G(k)}{k} \right) \frac{q_0 \cos^2 \Lambda}{2 V_c \cos \Lambda} \right) \frac{ac}{\cos \Lambda} \left( q_0 \cos^2 \Lambda + mg e_g \right) \\
    \frac{c_L a}{\cos \Lambda} + \frac{\epsilon_a}{k} \frac{q_0 \cos^2 \Lambda}{V_c \cos \Lambda}
\end{bmatrix}
$$

(61)

Note that $A$, $B$, and $w$ are dependent on the reduced frequency parameter $k$ which represents the effect of unsteady aerodynamics.

### III. Distributed Parameter Optimal Control of Aeroelastic Systems and Adjoint
Aerelastic Differential Operator

Having established a means to analyze stability of an aeroelastic system, we can now consider a distributed parameter control problem. We want to design a feedback controller $u = -K_x x$ that stabilizes any unstable aeroelastic modes. Optimal control theory can be used to accomplish a control design. Consider the following modal-weighted linear quadratic cost function

$$
J = \lim_{t_f \to \infty} \int_0^{t_f} \left[ \frac{1}{2} \langle x, Q x \rangle + \frac{1}{2} \langle u, R u \rangle \right] dt
$$

(62)

where $Q = Q^T > 0$ and $R = R^T > 0$ are weighting matrices, and the bracket $\langle \rangle$ denotes the inner product operation which is defined as

$$
\langle y, w \rangle = \int_0^L y^T w dy
$$

(63)

where $v$ and $w$ are any vectors of the same dimension.

The Lagrange multiplier method can be used to augment the cost function as

$$
J = \lim_{t_f \to \infty} \int_0^{t_f} \left[ \frac{1}{2} \langle x, Q x \rangle + \frac{1}{2} \langle u, R u \rangle + \langle \mu, Ax + Bu - \frac{\partial x}{\partial t} \rangle \right] dt
$$

(64)

where $\mu(t)$ is an adjoint state vector.

The Hamiltonian function is defined as

$$
H = \frac{1}{2} \langle x, Q x \rangle + \frac{1}{2} \langle u, R u \rangle + \langle \mu, Ax + Bu \rangle
$$

(65)

Let $\delta x$ and $\delta u$ be variations of $x$ and $u$, respectively. Then, the variation of the cost function is evaluated as

$$
\delta J = \lim_{t_f \to \infty} \int_0^{t_f} \left[ H \delta x + H_u \delta u - \langle \mu, \frac{\partial x}{\partial t} \rangle \right] dt
$$

(66)

Integrating by parts yields

$$
\delta J = \lim_{t_f \to \infty} \int_0^{t_f} \left[ H \delta x + H_u \delta u + \left( \frac{\partial u}{\partial t}, \delta x \right) - \langle \mu, (y, t_f), \delta x (y, t_f) \rangle + \langle \mu (y, 0), \delta x (y, 0) \rangle \right] dt
$$

(67)
The necessary conditions for optimality are obtained as

\[ H_x \delta x + \left\langle \frac{\partial \mu}{\partial t} , \delta x \right\rangle = 0 \quad (68) \]

\[ H_u = 0 \quad (69) \]

\[ \langle \mu (y, t_f) , \delta x (y, t_f) \rangle = 0 \quad (70) \]

\[ \langle \mu (y, 0) , \delta x (y, 0) \rangle = 0 \quad (71) \]

Consider Eq. (68) which can be expressed as

\[ \int_0^L \left[ \langle x, Q \delta x \rangle + \langle \mu, A \delta x \rangle + \left\langle \frac{\partial \mu}{\partial t}, \delta x \right\rangle \right] \, dy = 0 \quad (72) \]

We define an adjoint aeroelastic differential operator \( A^* \) endowed with the following inner product property

\[ \langle \mu, A \delta x \rangle = \langle \delta x, A^* \mu \rangle \quad (73) \]

Because of the symmetry of the inner product, it follows that \( A \) is also the adjoint operator of \( A^* \), that is, \( A = (A^*)^* \).

Equation (72) then becomes

\[ \int_0^L \left[ \langle \delta x, Qx \rangle + \langle \delta x, A^* \mu \rangle + \left\langle \delta x, \frac{\partial \mu}{\partial t} \right\rangle \right] \, dy = 0 \quad (74) \]

which can be written as

\[ \delta x H^*_x + \left\langle \delta x, \frac{\partial \mu}{\partial t} \right\rangle = 0 \quad (75) \]

where \( H^*_x \) is the adjoint of \( H_x \) which is defined as

\[ \delta x H^*_x = H_x \delta x \quad (76) \]

Therefore, the adjoint aeroelastic partial differential equation is obtained as

\[ \frac{\partial \mu}{\partial t} = -Qx - A^* \mu \quad (77) \]

Since \( x(y, 0) \) is given along with the boundary conditions, therefore \( \delta x(y, 0) = 0 \). Equation (116) is thus satisfied identically. Since \( x(y, t_f) \) is not known, therefore \( \delta x(y, t_f) \neq 0 \). Thus, in order for Eq. (112) to be satisfied, \( \mu(y, t_f) = 0 \) which is the transversality condition.

We will now evaluate the adjoint aeroelastic differential operator \( A^* \) for the aeroelastic problem of combined wing bending and torsion. Consider the following adjoint operation for torsion

\[ \left\langle v, \frac{\partial}{\partial y} \left( GJ \frac{\partial}{\partial y} \right) \right\rangle w = \left\langle \left[ \frac{\partial}{\partial y} \left( GJ \frac{\partial}{\partial y} \right) \right]^* \right\rangle v, w \quad (78) \]

with the associated boundary conditions \( w(0, t) = 0 \) and \( \frac{\partial w}{\partial y}(L, t) = 0 \).

Integration by parts yields

\[ \left\langle v, \frac{\partial}{\partial y} \left( GJ \frac{\partial}{\partial y} \right) \right\rangle w = v GJ \frac{\partial w}{\partial y} \bigg|_0^L - GJ \frac{\partial v}{\partial y} w \bigg|_0^L + \left\langle GJ \frac{\partial^2}{\partial y^2}, v, w \right\rangle \quad (79) \]

Therefore, the adjoint differential operator for torsion is obtained as

\[ \left[ \frac{\partial}{\partial y} \left( GJ \frac{\partial}{\partial y} \right) \right]^* = GJ \frac{\partial^2}{\partial y^2} \quad (80) \]

with the associated boundary conditions \( v(0, t) = 0 \) and \( \frac{\partial v}{\partial y}(L, t) = 0 \) when the adjoint differential operator is operated on \( v \).
Similarly, the adjoint operation for bending is given by

\[ \left\langle v, \frac{\partial^2}{\partial y^2} \left( EI \frac{\partial^2}{\partial y^2} \right) w \right\rangle = \left\langle \left[ \frac{\partial^2}{\partial y^2} \left( EI \frac{\partial^2}{\partial y^2} \right) \right]^* v, w \right\rangle \]  \tag{81}

with the associated boundary condition \( w(0,t) = 0, \frac{\partial w}{\partial y}(0,t) = 0, \frac{\partial^2 w}{\partial y^2}(L,t) = 0, \text{ and } \frac{\partial}{\partial y} \left( EI \frac{\partial^2 w}{\partial y^2} \right)(L,t) = 0. \)

Integration by parts yields

\[ \left\langle v, \frac{\partial^2}{\partial y^2} \left( EI \frac{\partial^2}{\partial y^2} \right) w \right\rangle = v \frac{\partial}{\partial y} \left( EI \frac{\partial^2 w}{\partial y^2} \right) \bigg|_{0}^{L} - \frac{\partial v}{\partial y} \frac{\partial^2 w}{\partial y^2} \bigg|_{0}^{L} + EI \frac{\partial^2 v}{\partial y^2} \frac{\partial w}{\partial y} \bigg|_{0}^{L} - EI \frac{\partial^3 v}{\partial y^3} w \bigg|_{0}^{L} + \left\langle EI \frac{\partial^4}{\partial y^4} v, w \right\rangle \]  \tag{82}

Thus, the adjoint differential operator for bending is obtained as

\[ \left[ \frac{\partial^2}{\partial y^2} \left( EI \frac{\partial^2}{\partial y^2} \right) \right]^* = EI \frac{\partial^4}{\partial y^4} \]  \tag{83}

with the associated boundary conditions \( v(0,t) = 0, \frac{\partial v}{\partial y}(0,t) = 0, \frac{\partial^2 v}{\partial y^2}(L,t) = 0, \text{ and } \frac{\partial^3 v}{\partial y^3}(L,t) = 0 \) when the adjoint differential operator is operated on \( v. \)

Also, the adjoint operation of the gradient is defined as

\[ \left\langle v, \frac{\partial}{\partial y} w \right\rangle = \left\langle \frac{\partial}{\partial y} v, w \right\rangle \]  \tag{84}

with the boundary condition \( w(0,t) = 0. \)

Then integration by parts yields

\[ \left\langle v, \frac{\partial}{\partial y} w \right\rangle = vw|_{0}^{L} + \left\langle - \frac{\partial}{\partial y} v, w \right\rangle \]  \tag{85}

Therefore

\[ \frac{\partial^*}{\partial y} = - \frac{\partial}{\partial y} \]  \tag{86}

with the boundary condition \( v(L,t) = 0. \)

Thus, the adjoint aeroelastic differential operator is obtained as

\[ A^* = \begin{bmatrix} 0 & -K^*M^{-T} \\ I & -C^*M^{-T} \end{bmatrix} \]  \tag{87}

where \( C^* \) and \( K^* \) are the adjoint damping and stiffness operators defined as

\[ C^* = \begin{bmatrix} \left[ e^{c_{la}F(k) \cos{\Lambda}} - \frac{e^{c_{la}G(k) \cos{\Lambda}}}{2} + \frac{\pi e_{la}^2 c_{la} \cos{\Lambda}}{\tan{\Lambda} k} \right] \frac{\partial^2}{\partial y^2} q_{w \cos{2\Lambda}}^2 & \left[ -e^{c_{la}F(k) \cos{\Lambda}} + \frac{e^{c_{la}G(k) \cos{\Lambda}}}{2} + \frac{\pi e_{la}^2 c_{la} \cos{\Lambda}}{\tan{\Lambda} k} \right] \frac{\partial^2}{\partial y^2} q_{w \cos{2\Lambda}}^2 \\ \frac{\partial}{\partial y} \left( GJ \frac{\partial}{\partial y} \right)^* - e^{c_{la}F(k) \cos{\Lambda}} q_{w \cos{2\Lambda}}^2 & c_{la} F(k) q_{w \cos{2\Lambda}}^2 \end{bmatrix} \]  \tag{88}

\[ K^* = \begin{bmatrix} \left[ -e^{c_{la}F(k) \cos{\Lambda}} \right]^* - e^{c_{la}F(k) \cos{\Lambda}} c_{la} F(k) q_{w \cos{2\Lambda}}^2 \frac{\partial^2}{\partial y^2} \left( EI \frac{\partial^2}{\partial y^2} \right)^* + c_{la} F(k) q_{w \cos{2\Lambda}}^2 \frac{\partial^2}{\partial y^2} \]  \tag{89}

The optimal control is obtained from

\[ \langle H_u, \delta u \rangle = \langle Ru + B^\top \mu, \delta u \rangle = 0 \]  \tag{90}

This results in

\[ u = -R^{-1}B^\top \mu \]  \tag{91}

Now consider a linear mapping of the adjoint state vector \( \mu \) onto the state vector \( x \) defined by \( \mu = Px \) where \( P \) is a linear operator. Substituting the linear mapping into the adjoint equation yields

\[ \frac{\partial P}{\partial t} x + P \left( Ax - BR^{-1}B^\top P x \right) = -Qx - A^*Px \]  \tag{92}
This results in the following differential operator Ricatti equation

\[
\frac{\partial P}{\partial t} + PA + A^*P - PBR^{-1}B^TP + Q = 0
\]  

(93)

For infinite time horizon optimal control, the differential operator Ricatti equation reduces to the algebraic operator Ricatti equation

\[
\mathcal{R} = PA + A^*P - PBR^{-1}B^TP + Q = 0
\]  

(94)

where \( \mathcal{R} \) is defined as the Ricatti operator.

Note that \( P \) is a self-adjoint operator since the adjoint operation on the Ricatti equation yields the same equation. Therefore, \( P = P^* \) and \( P \) has real eigenvalues.

\[
P = P^* = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} P^*_{11} & P^*_{12} \\ P^*_{21} & P^*_{22} \end{bmatrix}
\]  

(95)

IV. Approximation of Algebraic Operator Ricatti Equation

The solution of the algebraic operator Ricatti equation is non-trivial since it is a nonlinear operator equation. There is no existing tool available to solve this equation exactly. We will consider four approximate methods for solving the operator Ricatti equation.

A. Method 1: Distributed Parameter Spatially Continuous Optimal Control

The solution \( x(y,t) \) of the operator state-space equation can then be approximated by a finite series modal approximation

\[
x(y,t) = \begin{bmatrix} \Phi \\ W \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{N} \Phi_n(y) \theta_n(t) \\ \sum_{n=1}^{N} \Psi_n(y) \dot{w}_n(t) \end{bmatrix} = \sum_{n=1}^{N} \begin{bmatrix} \phi_n(y) & 0 & 0 & 0 \\ 0 & \psi_n(y) & 0 & 0 \\ 0 & 0 & \phi_n(y) & 0 \\ 0 & 0 & 0 & \psi_n(y) \end{bmatrix} \begin{bmatrix} \theta_n(t) \\ w_n(t) \\ \dot{\theta}_n(t) \\ \dot{w}_n(t) \end{bmatrix}
\]

\[
= \sum_{n=1}^{N} \Phi_n(y) z_n(t) = \Phi^T(y) z(t)
\]  

(96)

where \( N \) is the number of mode shapes, \( \theta_n(t) \) is a torsional generalized coordinate, \( w_n(t) \) is a bending generalized coordinate, \( z_n(t) \) is a vector of the generalized coordinates, \( \Phi_n(y) \) is a matrix of modal shape functions comprised of a torsional modal shape function \( \phi_n(y) \) and bending shape function \( \psi_n(y) \).

If \( N \rightarrow \infty \), then the infinite series solution of \( x(y,t) \) tends to the exact solution.

The generalized coordinates \( \theta_n(t) \) and \( w_n(t) \) are related to the modal generalized coordinate \( r_n(t) \) as follows:

\[
\begin{bmatrix} \theta_n(t) \\ w_n(t) \end{bmatrix} = \begin{bmatrix} c_\theta & 0 \\ c_w & 0 \end{bmatrix} \begin{bmatrix} r_n(t) \end{bmatrix}
\]  

(97)

\[
\begin{bmatrix} z_n(t) \end{bmatrix} = \begin{bmatrix} c_\theta & 0 \\ c_w & 0 \end{bmatrix} \begin{bmatrix} r_n(t) \end{bmatrix}
\]  

(98)

where \( c_\theta \) and \( c_w \) are some constant.

Therefore, \( x(y,t) \) can also be written as

\[
x(y,t) = \sum_{n=1}^{N} \begin{bmatrix} c_\theta \phi_n(y) & 0 & 0 & 0 \\ c_w \psi_n(y) & 0 & \phi_n(y) & 0 \\ 0 & \psi_n(y) & 0 & \end{bmatrix} \begin{bmatrix} r_n(t) \\ \dot{r}_n(t) \end{bmatrix} = \sum_{n=1}^{N} \Omega_n(y) p_n(t) = \Omega^Tv
\]  

(99)
Consider the eigenvalue problem

\[(\lambda - A)x = \begin{bmatrix} \sigma + i\omega & \sigma + i\omega + M^{-1}C \\ M^{-1}K & \sigma + i\omega + M^{-1}C \end{bmatrix} \begin{bmatrix} \Psi(y) r(t) \\ \Psi(y) \dot{r}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \] (100)

where \(\Psi(y) = \begin{bmatrix} c_\theta \phi_n(y) & c_w \psi_n(y) \end{bmatrix}^T\) is a modal shape vector function.

This results in

\[(M^{-1}K + \sigma^2 - \omega^2 + M^{-1}C)\Psi(y) r(t) = 0 \] (101)

\[(2\sigma\omega + M^{-1}C)\Psi(y) r(t) = 0 \] (102)

Thus, \(\Psi(y)\) is the eigenfunctions of the eigenvalue problem

\[M^{-1}C\Psi(y) = -2\sigma\Psi(y) \] (103)

\[M^{-1}K\Psi(y) = (\sigma^2 + \omega^2)\Psi(y) \] (104)

The aeroelastic differential operator \(A\) is then approximated by

\[Ax = \sum_{n=1}^{N} A_n \Omega_n(y) v_n(t) \] (105)

where \(A_n\) is an operator acting on the modal shape function \(\Psi_n(y)\)

\[A_n = \begin{bmatrix} 0 & I \\ -(\sigma_n^2 + \omega_n^2)I & 2\sigma I \end{bmatrix} \] (106)

\[\Omega_n(y) = \begin{bmatrix} \Psi_n(y) & 0 \\ 0 & \Psi_n(y) \end{bmatrix} \] (107)

The operator state-space equation is then expressed as

\[\sum_{n=1}^{N} \Omega_n(y) \dot{v}_n = \sum_{n=1}^{N} A_n \Omega_n(y) v_n + Bu + w \] (108)

Note that \(A_n\) is no longer a differential operator, but simply a matrix. The operator Ricatti equation then becomes a spatially dependent algebraic Ricatti equation

\[\mathcal{R}_n = P_n A_n + A_n^T P_n - P_n B R^{-1} B^T P_n + Q = 0 \] (109)

Note that since \(A_n\) is a matrix, then the adjoint operator \(A_n^*\) is equal to the transpose of \(A_n\).

The distributed parameter optimal control is then obtained as

\[u(y,t) = -R^{-1}B^T(y) \sum_{n=1}^{N} P_n(y) \Omega_n(y) v_n \] (110)

The distributed parameter optimal control can then be thought of as being a superposition of the optimal control for each individual mode. The spatially continuous feedback gain is computed as

\[K_{\infty}(y) = R^{-1}B^T(y) P_n(y) \Omega_n(y) \] (111)

B. Method 2: Stepwise Continuous Optimal Control Allocation from Spatially Continuous Optimal Control

The distributed parameter control variable \(u(y,t)\) is a continuously differentiable function in \(y\). The notion of a spatially continuous control variable may not be realizable due to physical limitations. Therefore, in practical applications, this
notion can be realized in a form of a piecewise or stepwise continuous control variable. Suppose the control variable is a step function with the following definition:

$$u(y, t) = \begin{cases} 
  u_1(t), & y_0 \leq y < y_1 \\
  u_2(t), & y_1 \leq y < y_2 \\
  \vdots & \\
  u_M(t), & y_{M-1} \leq y < y_M 
\end{cases}$$  \hspace{1cm} (112)

Let \( \tilde{u}(t) = \begin{bmatrix} u_1 & u_2 & \cdots & u_M \end{bmatrix}^\top \) and \( \tilde{B}(y) = \begin{bmatrix} B_1 & B_2 & \cdots & B_M \end{bmatrix} \) where \( B_i = B_i(y) \). Each of the control effector \( u_i \) can influence the control effectiveness over the entire wing span represented by \( B_i \). To maintain stepwise continuity, the individual control variables are constrained relative to each other. Thus, a relative constraint is imposed on the control variables.

$$|u_{i+1} - u_i| \leq \Delta$$  \hspace{1cm} (113)

where \( \Delta \) is the relative difference between any pair of the control variables to prescribe stepwise continuity.

The optimal control solution is designed such that this relative constraint is met implicitly by a desired choice of the weighting matrices \( Q \) and \( R \).

The spatially continuous distributed parameter optimal control creates a distributed control effectiveness vector \( M(y) = Bu \). The stepwise control must be designed so that it can achieve nearly the same control effectiveness as the spatially continuous control. This is effectively a control allocation problem. Toward this end, we formulate the following cost function

$$J = \int_0^{\tau_f} \|Bu - \tilde{B}\tilde{u}\|^2 dt$$  \hspace{1cm} (114)

Then, the stepwise continuous optimal control can be computed by computing the gradient of the cost function and setting it to zero. This results in

$$\frac{\partial J^\top}{\partial u} = -\int_0^L \tilde{B}^\top (Bu - \tilde{B}\tilde{u}) dy = 0$$  \hspace{1cm} (115)

The stepwise continuous optimal control is then obtained as

$$\tilde{u} = \left( \int_0^L \tilde{B}^\top \tilde{B} dy \right)^{-1} \int_0^L \tilde{B}^\top Bu dy$$  \hspace{1cm} (116)

C. Method 3: Optimal Control from Integral Operator State-Space

Another approach is to formulate an integral form of the operator state-space equation (108). Let \( A_\Omega = \begin{bmatrix} A_1\Omega_1 & \cdots & A_N\Omega_N \end{bmatrix} \). Then, the state-space equation in the matrix form is

$$\dot{\psi} = A_v \psi + B_v \tilde{u} + w_v$$  \hspace{1cm} (117)

where

$$A_v = \left( \int_0^L \Omega\Omega^\top dy \right)^{-1} \int_0^L \Omega A_\Omega dy$$  \hspace{1cm} (118)

$$B_v = \begin{bmatrix} 0 \\
  \left( \int_0^L \Psi M^\top \Psi dy \right)^{-1} \int_0^L \Psi F dy \\
  \left( \int_0^L \Psi M^\top \Psi dy \right)^{-1} \int_0^L \Psi F dy \\
  \left( \int_0^L \Psi M^\top \Psi dy \right)^{-1} \int_0^L \Psi F dy \\
  \left( \int_0^L \Psi M^\top \Psi dy \right)^{-1} \int_0^L \Psi F dy 
\end{bmatrix}$$  \hspace{1cm} (119)

$$w_v = \begin{bmatrix} 0 \\
  \left( \int_0^L \Psi M^\top \Psi dy \right)^{-1} \int_0^L \Psi F dy \\
  \left( \int_0^L \Psi M^\top \Psi dy \right)^{-1} \int_0^L \Psi F dy \\
  \left( \int_0^L \Psi M^\top \Psi dy \right)^{-1} \int_0^L \Psi F dy \\
  \left( \int_0^L \Psi M^\top \Psi dy \right)^{-1} \int_0^L \Psi F dy 
\end{bmatrix}$$  \hspace{1cm} (120)

Then the optimal control is then obtained as

$$\tilde{u} = -RB_v^\top \psi_v$$  \hspace{1cm} (121)

where \( P \) is the solution of the standard algebraic Riccati equation

$$PA_v + A_v^\top P - PB_vR^{-1}B_v^\top P + Q = 0$$  \hspace{1cm} (122)
D. Method 4: Optimal Control from Modal Transformation

A standard approach is to formulate the operator state-space model by a modal transformation directly from the discretized state-space model. Suppose this model is given in a standard state-space form

\[ \dot{x} = Ax + Bu \]  

(123)

where \( A \) and \( B \) are matrices of high dimensionality due to the discretization of an aeroelastic system by numerical methods such as the finite-element method.

The eigenmatrix of the \( A \) matrix contains the modal information of the aeroelastic system. A modal transformation can be applied to reduce the full state-space model into a reduced-order system in terms of the generalized coordinates of the modes of interest. Let \( v \) be a vector of the generalized coordinates. Then, using a finite series modal approximation, the equation of the aeroelastic system in the generalized coordinates is then obtained as

\[ \Psi M \Psi^T \ddot{r} + \Psi C \Psi^T \dot{r} + \Psi K \Psi^T r = \Psi F_{\delta} u + \Psi F \]  

(124)

This equation is also expressed in a state-space form as

\[ \dot{v} = A_g v + B_g u + w_g \]  

(125)

where

\[ A_g = \begin{bmatrix} 0 \\ - (\Psi M \Psi^T)^{-1} (\Psi K \Psi^T) \\ - (\Psi M \Psi^T)^{-1} (\Psi C \Psi^T) \end{bmatrix} \]  

(126)

\[ B_g = \begin{bmatrix} 0 \\ (\Psi M \Psi^T)^{-1} (\Psi F_{\delta}) \end{bmatrix} \]  

(127)

\[ w_g = \begin{bmatrix} 0 \\ (\Psi M \Psi^T)^{-1} (\Psi F) \end{bmatrix} \]  

(128)

Then the optimal control can also be computed by a standard linear-quadratic optimal control as

\[ \ddot{u} = - R^{-1} B_g P v \]  

(129)

where \( P \) is the solution of the standard algebraic Ricatti equation

\[ PA_g + A_g^T P - PB_g R^{-1} B_g^T P + Q = 0 \]  

(130)

Since the finite-series modal approximation is a weak-form expression of the governing strong-form partial differential equation, the notion of distributed parameter optimal control is no longer preserved. Therefore, it may not be possible to obtain a spatially continuous optimal control solution. On the other hand, a stepwise continuous optimal control solution may be attainable using the methods 2 and 3 above. The methods 3 and 4 may appear to be similar, but method 3 is derived from the operator state-space equation whereas method 4 is a standard approach where the mass, damping, and stiffness matrices are available from a discretization.

V. Flutter Suppression Control of a Flexible Wing Aircraft

Under the Fundamental Aeronautics Program at the NASA Aeronautics Research Mission Directorate, the Fixed Wing project is conducting multidisciplinary foundational research to investigate advanced concepts and technologies for future aircraft systems. A NASA study entitled “Elastically Shape Future Air Vehicle Concept” was conducted in 2010\(^{4,5} \) to examine new concepts that can enable active control of wing aeroelasticity to achieve drag reduction. This study showed that highly flexible wing aerodynamic surfaces can be elastically shaped in-flight by active control of wing twist and vertical deflection in order to optimize the local angle of attack of wing sections to improve aerodynamic efficiency through drag reduction during cruise and enhanced lift performance during take-off and landing.

The study shows that active aeroelastic wing shaping control can have a potential drag reduction benefit. Conventional flap and slat devices inherently generate drag as they increase lift. The study shows that conventional flap and slat systems are not aerodynamically efficient for use in active aeroelastic wing shaping control for drag reduction.
A new flap concept called the Variable Camber Continuous Trailing Edge Flap (VCCTEF) system was developed by NASA to address this need. Initial results indicate that the VCCTEF system may offer a potential pay-off for drag reduction that will result in significant fuel savings. In order to realize the potential benefit of drag reduction by active aeroelastic wing shaping control, configuration changes in high-lift devices have to be a part of the wing shaping control strategy.

NASA and Boeing are currently conducting a joint study to develop the VCCTEF further under the research element Active Aeroelastic Shape Control (AASC) within the Fixed Wing project. This study built upon the development of the VCCTEF system for NASA Generic Transport Model (GTM) which is essentially based on the B757 airframe, as shown in Fig. 2. The VCCTEF employs light-weight shaped memory alloy (SMA) technology for actuation and three separate chordwise segments shaped to provide a variable camber to the flap, as shown in Fig. 3. This cambered flap has potential for drag reduction as compared to a conventional straight, plain flap. The flap is also made up of individual 2-foot spanwise sections which enable different flap setting at each flap spanwise position. This results in the ability to control the wing twist shape as a function of span, resulting in a change to the wing twist to establish the best lift-to-drag ratio (L/D) at any aircraft gross weight or mission segment. Current wing twist on commercial transports is permanently set for one cruise configuration, usually for a 50% loading or mid-point on the gross weight schedule. The VCCTEF offers different wing twist settings for each gross weight condition and also different settings for climb, cruise and descent, a major factor in obtaining best L/D conditions.

The second feature of VCCTEF is a continuous trailing edge flap, as shown in Fig. 4. The individual 2-foot spanwise flap sections are connected with a flexible covering, so no breaks can occur in the flap platforms, thus reducing drag by eliminating these breaks in the flap continuity which otherwise would generate vorticity that results in a drag increase and also contributes to airframe noise. This continuous trailing edge flap design combined with the flap camber result in lower drag increase during flap deflections. In addition, it also offers a potential noise reduction benefit.

The VCCTEF serves multiple functions as:

- A wing shaping control device to twist the flexible wing to obtain changes in lift-to-drag ratios that will reduce cruise drag throughout the flight envelope using all three chordwise camber segments and spanwise segments to shape spanwise lift distribution.
- A high-lift device for take-off, climb-out, let-down and final approach by using the full span cambered flap.
- A full span roll control effector in lieu of traditional ailerons using the full span third camber aft segments of the cambered flap.
- An aeroservoelastic (ASE) control device to compensate for reduced flutter margins of flexible wings using the full span third camber aft segments of the cambered flap.

The first and second camber segments are actuated by SMA actuators which have a low frequency response and therefore are well suited for drag minimization control. The third camber segments are actuated by electric drive motors which have a faster frequency response for roll control and ASE control such as flutter suppression and gust load alleviation control.

![Figure 2 - GTM with VCCTEF](image)
In order to enable effective aeroelastic wing shaping control for drag reduction, the wing structures are designed to be highly flexible with the stiffness of the wing reduced by half. This stiffness results in a vertical wing tip deflection of about 10% of the wing semi-span, which is comparable to the wing tip deflection of Boeing 787 aircraft. Because the wings are highly flexible, flutter can become a potential issue.

A finite-element model is constructed to model the wing aeroelasticity, as shown in Fig. 5. This finite-element model is used to perform a flutter analysis. The critical flutter speed is determined to be associated with anti-symmetric modes. Figures 6 and 7 show the frequencies and critical damping ratios of the first five anti-symmetric modes as functions of the Mach number at 35,000 ft. The first critical flutter mode is due to the first bending mode (mode 1B) which occurs at 0.9282 when the critical damping ratio becomes negative. The next two critical modes are also nearby and are due to the third bending mode (3B) and first torsion mode (1T).
Fig. 6 - Frequencies of Anti-Symmetric Modes vs. Mach @ 35,000 ft

Fig. 7 - Critical Damping Ratios of Anti-Symmetric Modes vs. Mach @ 35,000 ft

The bending and torsion modal shape functions of the first four anti-symmetric aeroelastic modes are plotted in Figs. 8 and 9. The first three modes are associated with bending and the fourth mode is a torsion mode.
Figure 10 shows the critical flutter speed due to the first bending mode (mode 1B) as a function of the wing torsional stiffness. The results show a significant reduction in the flutter speed as the torsional stiffness decreases. The dive speed of the aircraft is assumed to be at Mach 0.96 at 35,000 ft. The FAA flutter clearance requires a minimum flutter speed 15% above the dive speed. This results in a minimum flutter speed of Mach 1.104. For a wing with half flexural and torsional stiffness, the first critical flutter speed occurs at Mach 0.9282 which is well below the FAA flutter speed requirement. Therefore, flutter suppression control can be considered as a possibility to increase the flutter margin.
The control effectiveness of the VCCTEF is specified by the lift and pitch moment derivatives of the third camber segments as shown in Figs. 11 and 12.
To design a flutter suppression control, the first five aeroelastic modes, not including the rigid body mode, are used for feedback. The finite-element model contains a total of 200 states. The eigenvalues and eigenvectors of the aeroelastic plant are computed by the \( p-k \) method whereby the reduced frequency is determined by an iterative method to satisfy the dependency of the plant on the reduced frequency. The \( A \) and \( B \) matrices are then computed and are used for the eigenvalue analysis.

In the method 1, the \( B \) matrix is a spatially dependent matrix discretized by 100 points along the wing span. Each of the 15 control surfaces contributes to the \( B \) matrix and are treated as a spatially continuous control that spans the length of the wing from the side of body of the aircraft. So, the dimension of the \( B \) matrix is 4 by 15 by 100. At each value of \( y \), a stabilizing controller is computed by the LQR method using Eq. (109) for each of the five modes. The weighting matrices \( Q = 10^{-2}I \) and \( R = 10^{-2}I \) are selected. The flight condition is Mach 0.90 at 35,000 ft. The resulting spatially continuous feedback gains \( K_x(n)(y) \) for the five modes are shown for the innermost flap 1, mid-span flap 8, and outermost flap 15. The spatially continuous feedback gains are converted into discrete feedback gains for the 15 stepwise continuous control surfaces. These are shown in Fig. 14. Note that both flap 1 and flap 15 are the most active control surfaces. In terms of meeting the relative constraint of any pair of adjacent flaps, flap 1 potentially could have a large deflection relative to flap 2 which may exceed the relative constraint.
The open-loop and closed-loop poles for the spatially continuous and stepwise continuous control are shown in Table 1. Without flutter suppression control, both aeroelastic modes 1B and 3B are unstable. With flutter suppression control using both types of continuous control realization, the unstable aeroelastic modes are stabilized.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Open-Loop Poles</th>
<th>Closed-Loop Poles Spatially Continuous</th>
<th>Closed-Loop Poles Stepwise Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>1B</td>
<td>$0.7654 \pm 18.4921i$</td>
<td>$-1.7320 \pm 18.3596i$</td>
<td>$-6.6062 \pm 16.9386i$</td>
</tr>
<tr>
<td>2B</td>
<td>$-3.6304 \pm 20.0728i$</td>
<td>$-3.5659 \pm 20.1013i$</td>
<td>$-3.8606 \pm 20.2372i$</td>
</tr>
<tr>
<td>3B</td>
<td>$1.8656 \pm 37.8924i$</td>
<td>$-2.8347 \pm 37.8355i$</td>
<td>$-1.6366 \pm 37.9827i$</td>
</tr>
<tr>
<td>1T</td>
<td>$-9.2478 \pm 40.8921i$</td>
<td>$-13.6664 \pm 39.7861i$</td>
<td>$-9.4120 \pm 41.1016i$</td>
</tr>
<tr>
<td>4B</td>
<td>$-0.3084 \pm 69.4515i$</td>
<td>$-1.7361 \pm 69.3891i$</td>
<td>$-0.3132 \pm 69.4553i$</td>
</tr>
</tbody>
</table>

Table 1 - Poles of Aeroelastic Modes at Mach 0.95

The integral operator method is used to compute the stepwise continuous control. The operator state-space equation is converted into an integral form. The optimal control is computed from the Ricatti equation (122). The weighting matrices $Q = 10^{-2}I$ and $R = 10^4I$ are selected. The discrete feedback gains for the 15 stepwise continuous control surfaces are plotted in Fig. 15. The discrete feedback gains have a distinct feature as compared to the stepwise continuous control in Fig. 14. These feedback gains exhibit a spatial variation that appears to follow
Fig. 15 - Stepwise Continuous Feedback Gain Distribution from Integral Operator Method

To compare with the standard modal transformation method, a state-space equation in terms of the generalized coordinates is constructed. The optimal control is computed from the Ricatti equation (130). The weighting matrices $Q = 10^{-3}I$ and $R = 10^3$ are selected. The discrete feedback gains for the 15 control surfaces are plotted in Fig. 16. The control action produced by the modal transformation method is significantly different from both the stepwise continuous and the integral operator methods. The control action is distributed throughout the 15 control surfaces. The discrete feedback gains for all the 15 control surfaces have the same sign for each of the five modes.
Fig. 16 - Feedback Gain Distribution from Standard LQR Modal Transformation Method

The critical damping ratios of the first five anti-symmetric modes as functions of the Mach number at 35,000 ft with flutter suppression control using the standard modal transformation is shown in Fig. 17.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Open-Loop Poles</th>
<th>Closed-Loop Poles Integral Operator</th>
<th>Closed-Loop Poles Modal Transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1B</td>
<td>0.7654 ± 18.4921i</td>
<td>−8.8693 ± 17.4467ii</td>
<td>−7.2530 ± 16.6401i</td>
</tr>
<tr>
<td>2B</td>
<td>−3.6304 ± 20.0728i</td>
<td>−2.8702 ± 19.1102i</td>
<td>−2.7543 ± 20.2796i</td>
</tr>
<tr>
<td>3B</td>
<td>1.8656 ± 37.8924i</td>
<td>−3.2372 ± 37.7543i</td>
<td>−2.0869 ± 36.7471i</td>
</tr>
<tr>
<td>1T</td>
<td>−9.2478 ± 40.8921i</td>
<td>−9.5193 ± 40.8270i</td>
<td>−7.6360 ± 40.9560i</td>
</tr>
<tr>
<td>4B</td>
<td>−0.3084 ± 69.4515i</td>
<td>−1.1393 ± 69.4426i</td>
<td>−0.6106 ± 69.3137i</td>
</tr>
</tbody>
</table>

Table 2 - Poles of Aeroelastic Modes at Mach 0.95
VI. Conclusions

This paper presents a distributed parameter optimal control method. The state-space equation of wing bending-torsion aeroelasticity is formulated using aeroelastic differential operators to describe the damping and stiffness matrices. The optimal control problem is formulated as a linear quadratic regulator in a Hilbert inner product space. The solution to the optimal control problem leads to a differential operator Ricatti equation. This differential operator Ricatti equation is solved using a finite-series modal approximation to convert the differential operator Ricatti equation into a spatially dependent Ricatti equation. The solution of this spatially dependent Ricatti equation is used to compute the distributed parameter optimal control which is spatially continuous. A stepwise continuous optimal control can be obtained from the spatially continuous optimal control using a control allocation technique. An alternative method based on an integral operator formulation is presented. A flutter suppression control application for a flexible wing generic transport model demonstrates the effectiveness of the distributed parameter optimal control approaches as compared to the standard optimal control using the modal transformation for aeroelastic systems.

References

2Theodorsen, T., Garrick, I., “Mechanism of Flutter - a Theoretical and Experimental Investigation of the Flutter Problem”, NACA Report 685, 1940.