Abstract Interpretation: a methodology for the rapid development of provably correct static analyzers

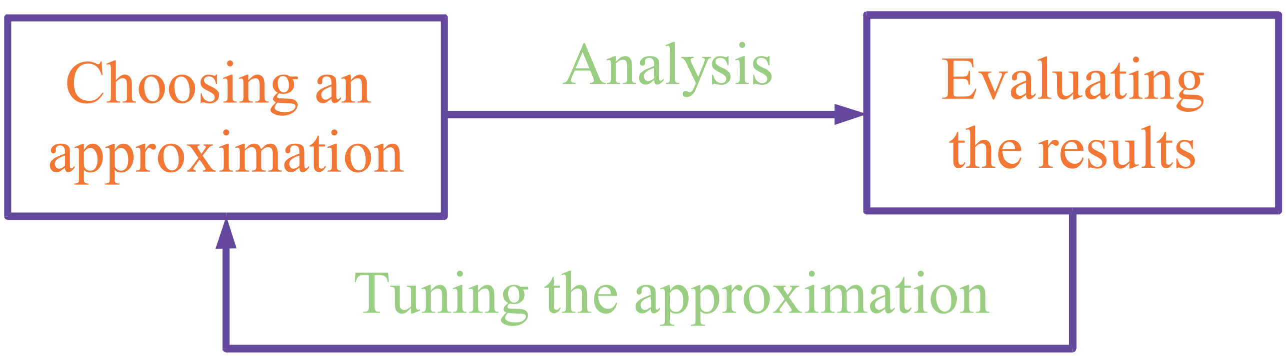
Arnaud Venet

Kestrel Technology
NASA Ames Research Center

arnaud@email.arc.nasa.gov

Static analysis in real life

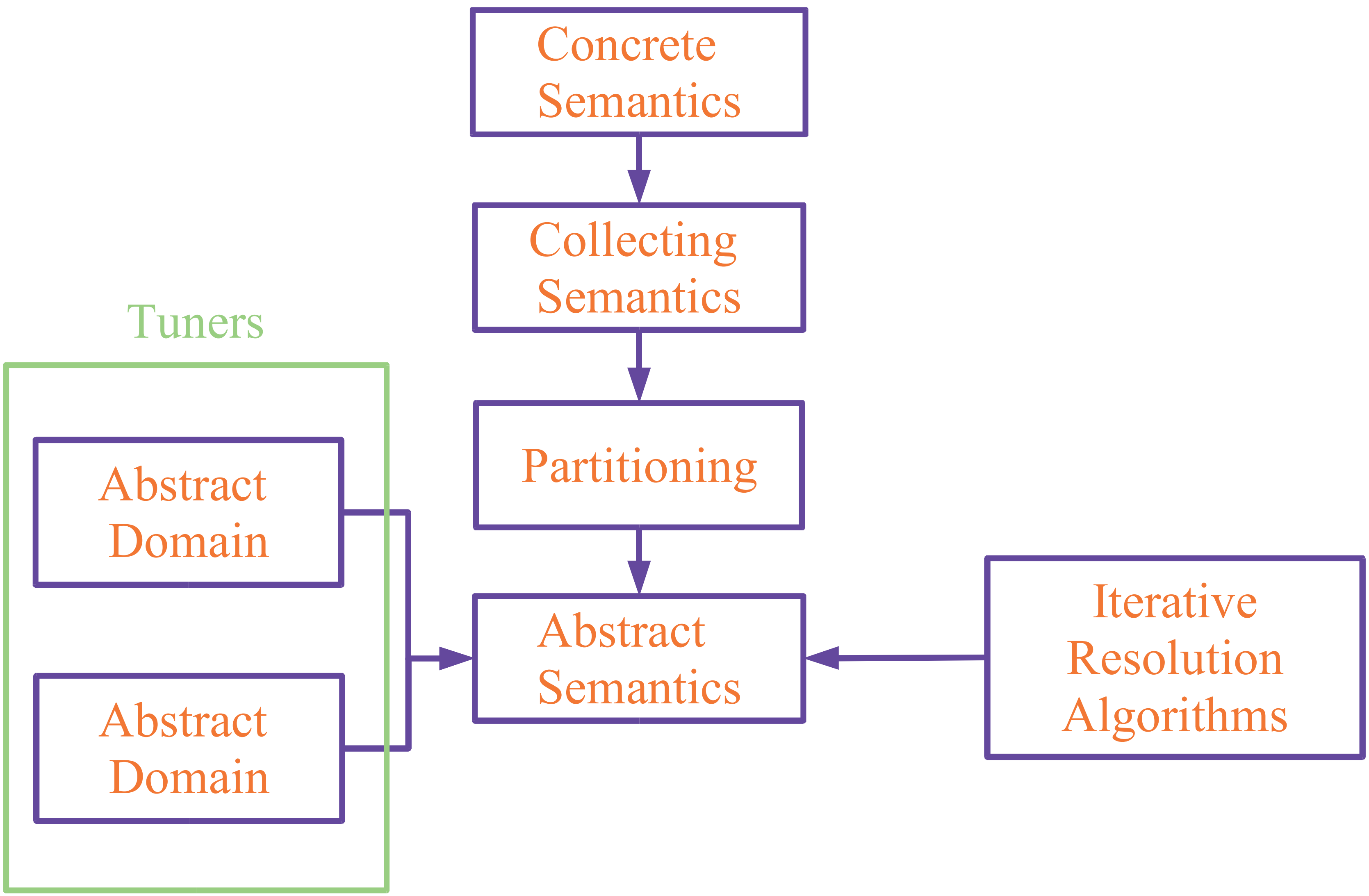
- Undecidable problem: automatic program verification $\Rightarrow$ **loops**
- Approximation for decidability: false positives
- Tradeoff precision/efficiency
- The approximation should be tunable:



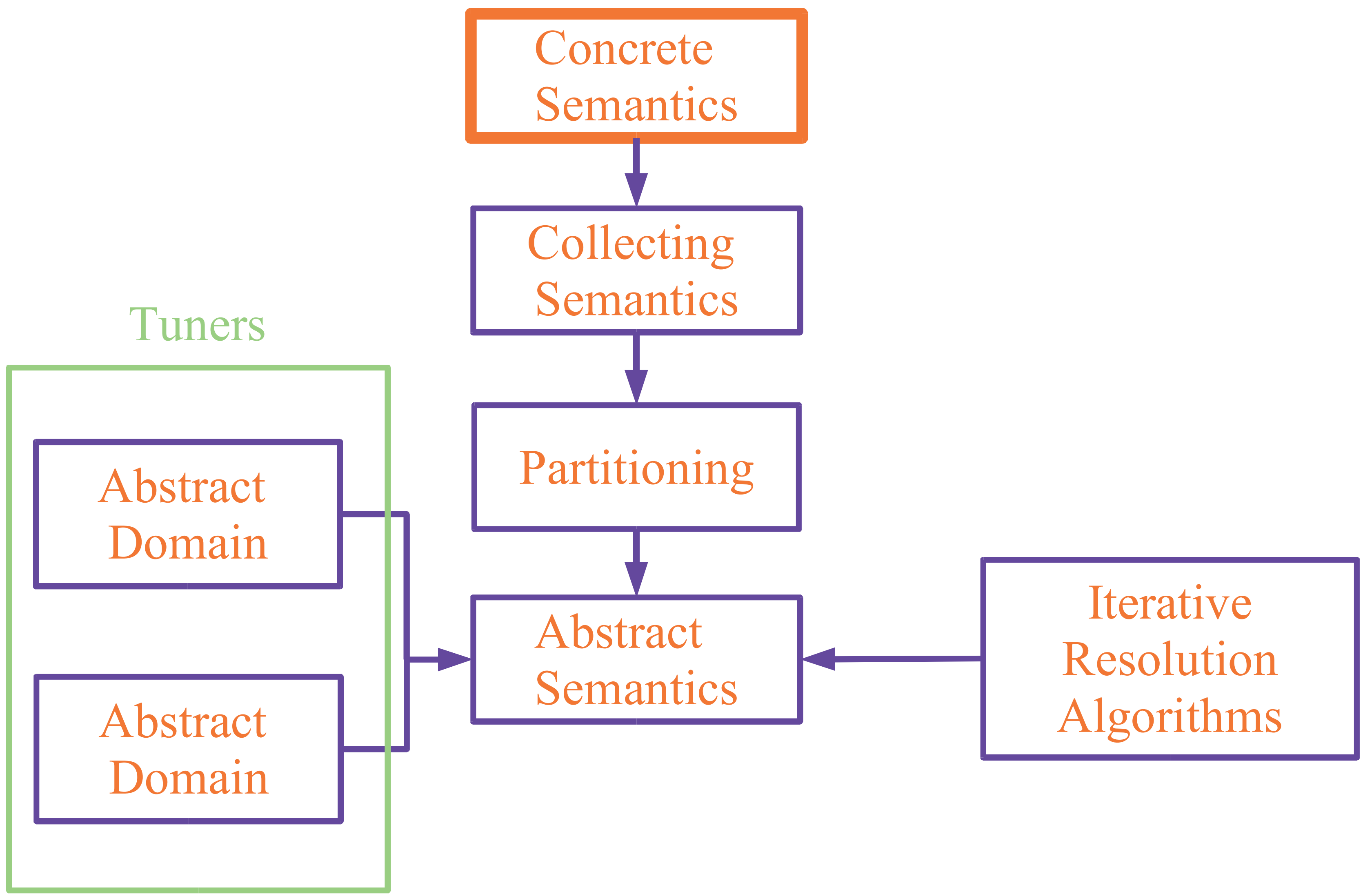
Abstract Interpretation

+ A general methodology for building static analyzers
+ Provides generic algorithms
+ Approximation and resolution are separated: the analyzers are tunable by construction
+ The soundness proof goes along with the analyzer design
- Scalability is difficult to achieve

Methodology



Methodology



Concrete semantics

Small-step operational semantics: $(\Sigma, \rightarrow)$

$s = \langle \text{program point}, \text{env} \rangle$

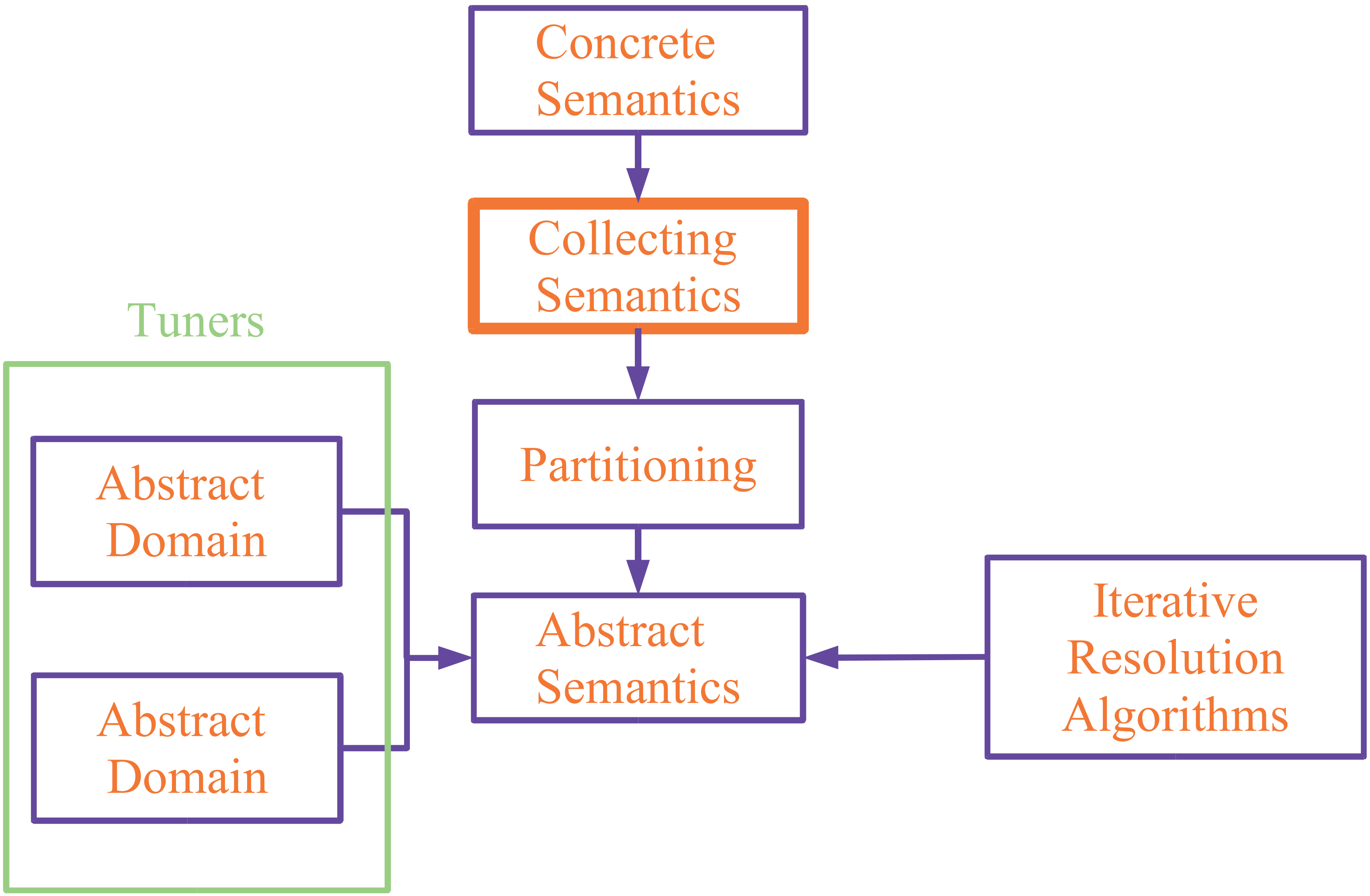
$s \rightarrow s'$

<u>Example:</u>

```
1:   n = 0;
2:   while n < 1000 do
3:     n = n + 1;
4:   end
5:   exit
```

$$\langle 1, n \Rightarrow \Omega \rangle \rightarrow \langle 2, n \Rightarrow 0 \rangle \rightarrow \langle 3, n \Rightarrow 0 \rangle \rightarrow \langle 4, n \Rightarrow 1 \rangle$$
$$\rightarrow \langle 2, n \Rightarrow 1 \rangle \rightarrow \ldots \rightarrow \langle 5, n \Rightarrow 1000 \rangle$$

Methodology



Collecting semantics

The first abstraction step. It defines the observable behaviors of programs:

- Sets of states (e.g. range of variables)
- Sets of finite traces (e.g. computational dependencies)
- Sets of finite and infinite traces (e.g. termination properties)

State properties

The set of descendants of the initial state s_0:

$$\boldsymbol{S} = \{s \mid s_0 \to \ldots \to s\}$$

<u>Theorem:</u> $\boldsymbol{F} : (\wp(\Sigma), \subseteq) \to (\wp(\Sigma), \subseteq)$

$$\boldsymbol{F}(S) = \{s_0\} \cup \{s' \mid \exists s \in S: s \to s'\}$$

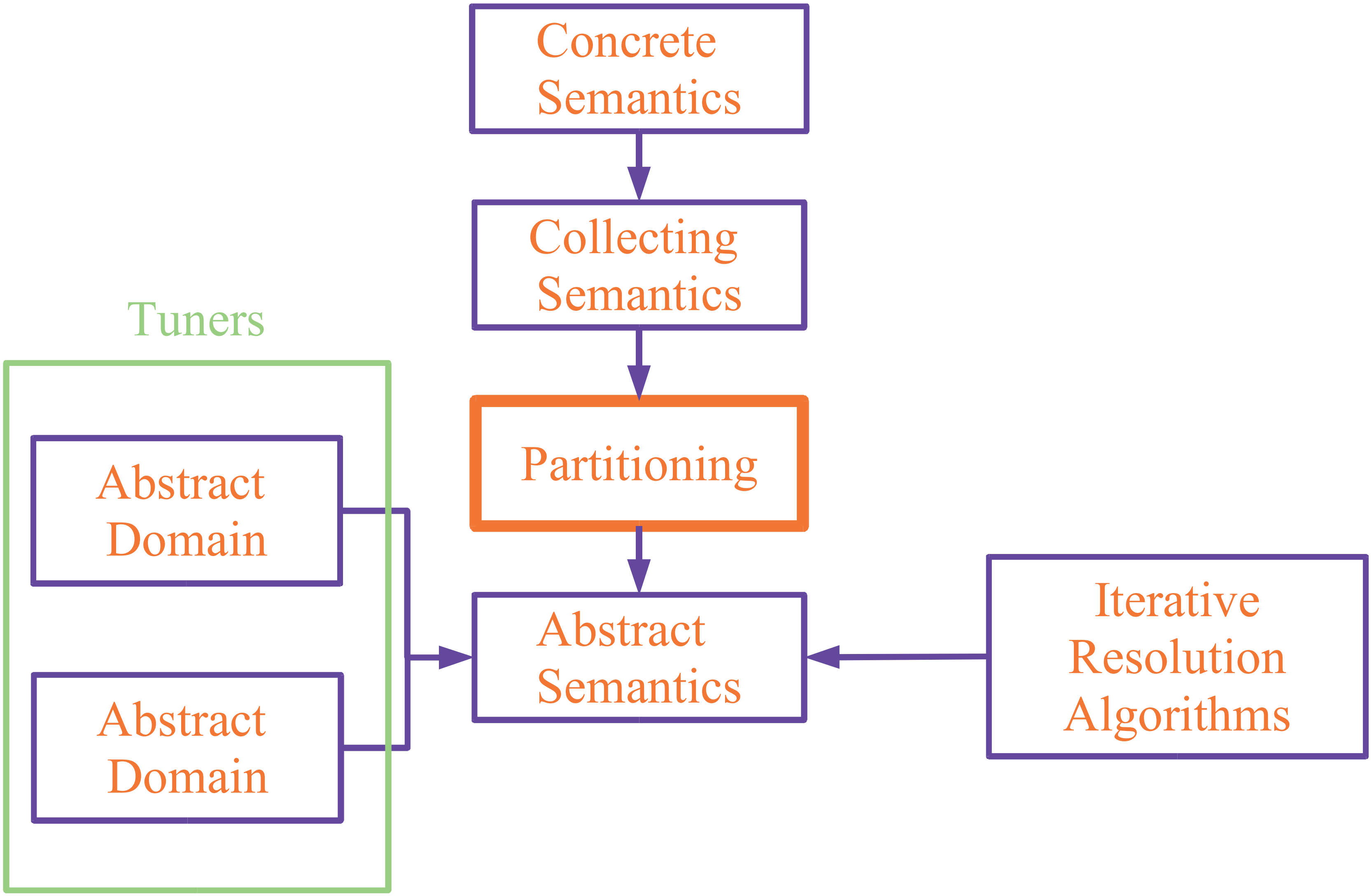
$$\boldsymbol{S} = \mathrm{lfp}\ \boldsymbol{F}$$

Example

```
1:   n = 0;
2:   while n < 1000 do
3:     n = n + 1;
4:   end
5:   exit
```

$$\boldsymbol{S} = \{\langle 1, n \Rightarrow \Omega\rangle, \langle 2, n \Rightarrow 0\rangle, \langle 3, n \Rightarrow 0\rangle, \langle 4, n \Rightarrow 1\rangle, \langle 2, n \Rightarrow 1\rangle, \ldots, \langle 5, n \Rightarrow 1000\rangle\}$$

Methodology



Partitioning

We partition the set Σ of states w.r.t. program points:

- $\Sigma = \Sigma_1 \oplus \Sigma_2 \oplus \dots \oplus \Sigma_n$
- $\mathcal{F}(S_1, \dots, S_n)_i = \{s' \in S_i \mid \exists j\ \exists s \in S_j : s \to s'\}$
- Control-flow graph: $(\mathcal{P}, \boldsymbol{\rightarrow})$
- $\mathcal{F}(S_1, \dots, S_n)_i = \{\langle i, \varepsilon' \rangle \mid \exists j \boldsymbol{\rightarrow} i : \langle j, \varepsilon \rangle \to \langle i, \varepsilon' \rangle\}$

Semantic equations

- $i \rightarrow j$: operation **op**
- Notation: E_i = set of environments at program point i
- $[\mathbf{op}]\varepsilon$ = semantics of **op**
- System of semantic equations:

$$E_i = \bigcup \{[\mathbf{op}]E_j \mid j \rightarrow i : \mathbf{op}\}$$

- Solution of the system = $\boldsymbol{S}$ = lfp $\boldsymbol{F}$

Example

```
n = 0;
while n < 1000 do
  n = n + 1;
end
exit
```

$$E_1 = \{n \Rightarrow \Omega\}$$

$$E_2 = [\texttt{n = 0}]E_1 \cup E_4$$

$$E_3 = E_2 \cap\,]-\infty, 999]$$

$$E_4 = [\texttt{n = n + 1}]E_3$$

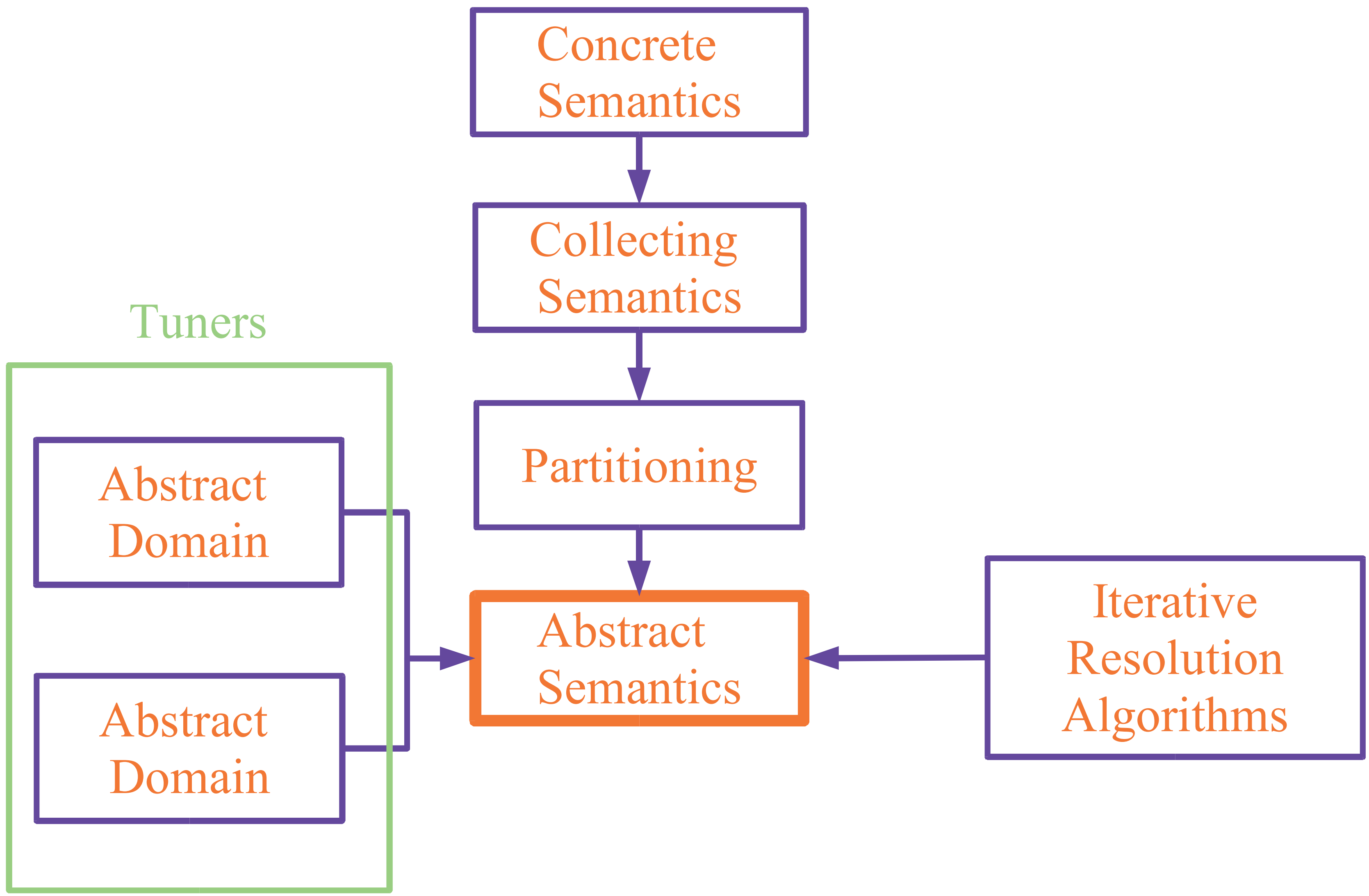
$$E_5 = E_2 \cap [1000, +\infty[$$

Other kinds of partitioning

In the case of collecting semantics of traces:

- Partitioning w.r.t. procedure calls: **context sensitivity**
- Partitioning w.r.t. executions paths in a procedure: **path sensitivity**
- Dynamic partitioning (Bourdoncle)

Methodology



Approximation

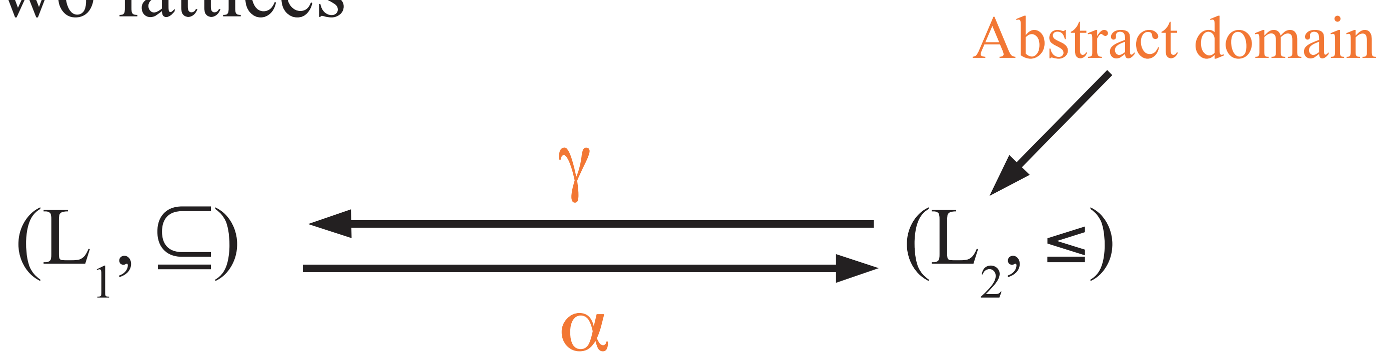
Problem: Compute a sound approximation $s^\#$ of s

$$s \subseteq s^\#$$

Solution: Galois connections

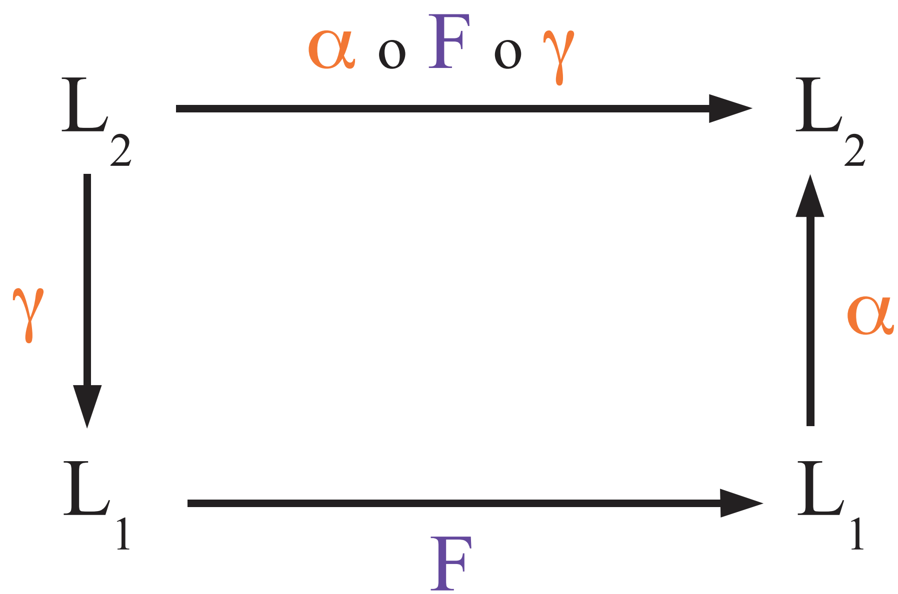
Galois connection

L_1, L_2 two lattices



- $\forall x \forall y : \alpha(x) \leq y \Leftrightarrow x \subseteq \gamma(y)$
- $\forall x \forall y : x \subseteq \gamma \circ \alpha(x)$ & $\alpha \circ \gamma(y) \leq y$

Fixpoint approximation



<u>Theorem:</u>

$$\text{lfp } F \subseteq \gamma\,(\text{lfp } \alpha \circ F \circ \gamma)$$

Abstracting the collecting semantics

- Find a Galois connection:

$$(\wp(\Sigma), \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (\Sigma^{\#}, \leq)$$

- Find a function: $\alpha \circ \boldsymbol{F} \circ \gamma \leq \boldsymbol{F}^{\#}$

Partitioning ⇨ Abstract sets of environments

Abstract algebra

- Notation: $\boldsymbol{E}$ the set of all environments
- Galois connection:

$$(\wp(\boldsymbol{E}), \subseteq) \underset{\alpha}{\overset{\gamma}{\rightleftarrows}} (\boldsymbol{E}^{\#}, \leq)$$

- $\cup, \cap$ approximated by $\cup^{\#}, \cap^{\#}$
- $[\mathbf{op}]$ approximated by $[\mathbf{op}]^{\#}$

$$\alpha \circ [\mathbf{op}] \circ \gamma \;\leq\; [\mathbf{op}]^{\#}$$

Abstract semantic equations

```
1:   n = 0;
2:   while n < 1000 do
3:     n = n + 1;
4:   end
5:   exit
```

$$E_1^{\#} = \alpha(\{n \Rightarrow \Omega\})$$

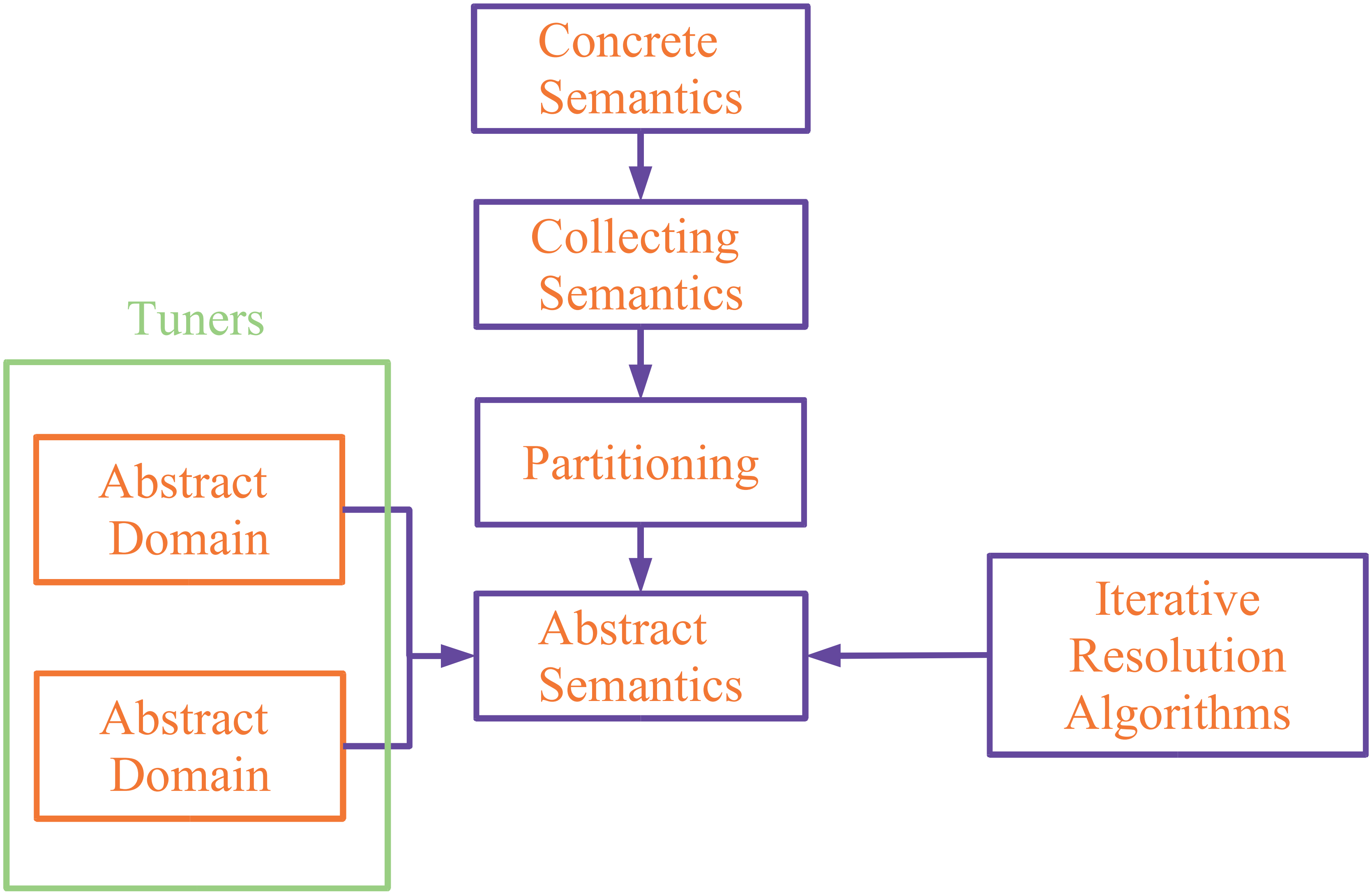
$$E_2^{\#} = [\texttt{n = 0}]^{\#} E_1^{\#} \cup^{\#} E_4^{\#}$$

$$E_3^{\#} = E_2^{\#} \cap^{\#} \alpha(]-\infty, 999])$$

$$E_4^{\#} = [\texttt{n = n + 1}]^{\#} E_3^{\#}$$

$$E_5^{\#} = E_2^{\#} \cap^{\#} \alpha([1000, +\infty[)$$

Methodology



Abstract domains

Environment: $x \Rightarrow v$, $y \Rightarrow w$, ...

Various kinds of approximations:

- Intervals (nonrelational):

$$x \Rightarrow [a, b],\ y \Rightarrow [a', b'],\ \ldots$$

- Polyhedra (relational):

$$x + y - 2z \leq 10,\ \ldots$$

- Difference-bound matrices (weakly relational):

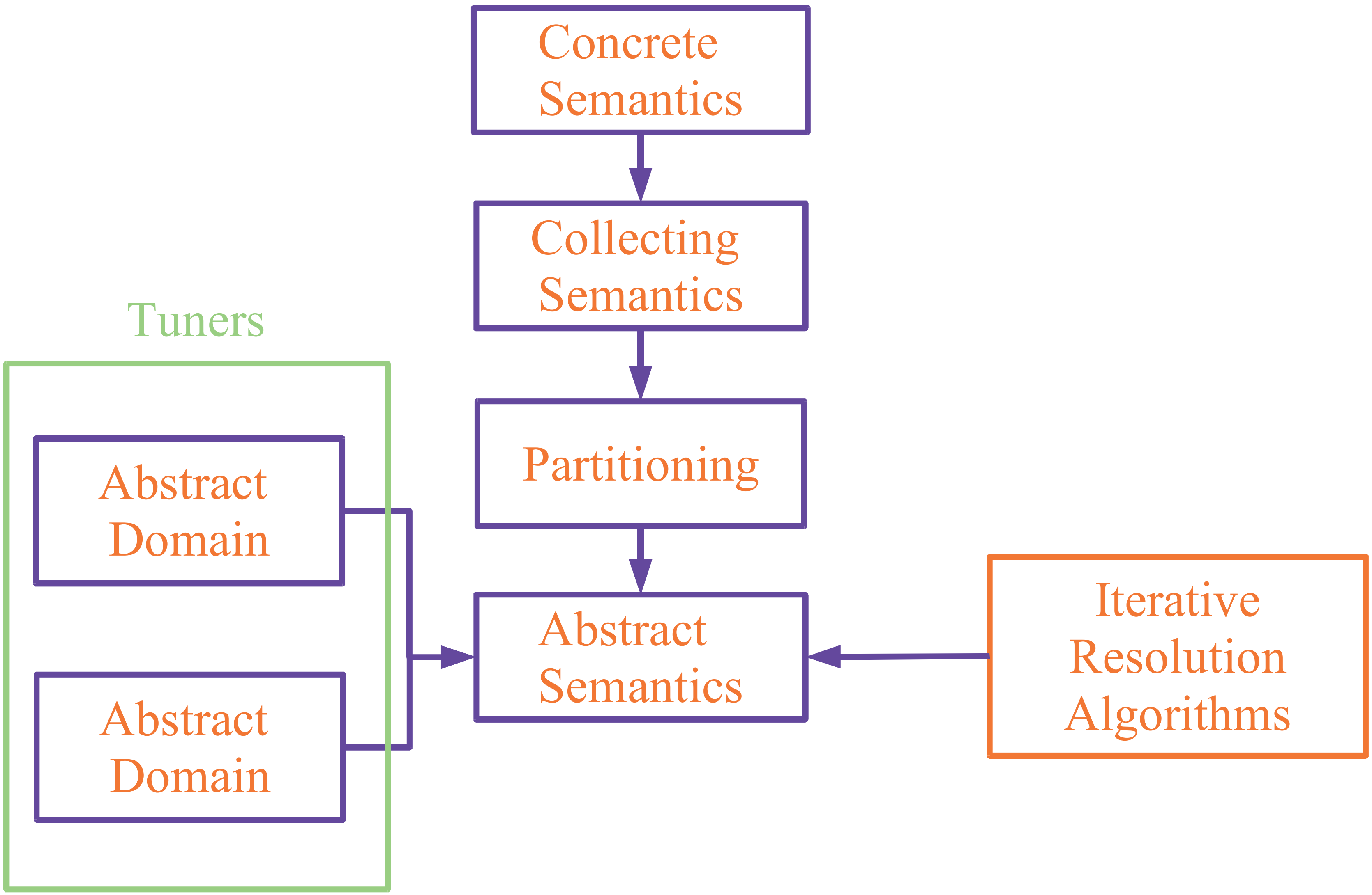
$$y - x \leq 5,\ z - y \leq 10,\ \ldots$$

Example: intervals

```
n = 0;
while n < 1000 do
  n = n + 1;
end
exit
```

- Iteration 1: $E_2^{\#} = [0, 0]$
- Iteration 2: $E_2^{\#} = [0, 1]$
- Iteration 3: $E_2^{\#} = [0, 2]$
- Iteration 4: $E_2^{\#} = [0, 3]$
- ...

Methodology



Widening operator

Lattice $(L, \leq)$: $\nabla : L \times L \to L$

- Abstract union operator:

$$\forall x \forall y : \; x \leq x \nabla y \;\; \& \;\; y \leq x \nabla y$$

- Enforces convergence: $(x_n)_{n \geq 0}$

$$\begin{cases} y_0 = x_0 \\ y_{n+1} = y_n \nabla x_{n+1} \end{cases}$$

$(y_n)_{n \geq 0}$ is ultimately stationary

Widening of intervals

$$[a, b] \nabla [a', b']$$

- If $a \leq a'$ then a else $-\infty$
- If $b' \leq b$ then b else $+\infty$

➥ Open unstable bounds (jump over the fixpoint)

Iteration with widening

```
1:   n = 0;
2:   while n < 1000 do
3:     n = n + 1;
4:   end
5:   exit
```

$$(E_2^{\#})_{n+1} = (E_2^{\#})_n \,\nabla\, \Big([\mathtt{n = 0}]^{\#}(E_1^{\#})_n \cup^{\#} (E_4^{\#})_n\Big)$$

Iteration 1 (union): $E_2^{\#} = [0, 0]$

Iteration 2 (union): $E_2^{\#} = [0, 1]$

Iteration 3 (widening): $E_2^{\#} = [0, +\infty] \Rightarrow$ stable

Imprecision at loop exit

```
1:   n = 0;
2:   while n < 1000 do
3:     n = n + 1;
4:   end
5:   exit; t[n] = 0;
```

- $E_5^{\#} = [1000, +\infty[$
- The information is present in the equations

Narrowing operator

Lattice $(L, \leq)$: $\Delta : L \times L \to L$

- Abstract intersection operator:

$$\forall x \forall y : x \cap y \leq x \, \Delta \, y$$

- Enforces convergence: $(x_n)_{n \geq 0}$

$$\begin{cases} y_0 = x_0 \\ y_{n+1} = y_n \, \Delta \, x_{n+1} \end{cases}$$

$(y_n)_{n \geq 0}$ is ultimately stationary

Narrowing of intervals

$$[a, b] \Delta [a', b']$$

- If $a = -\infty$ then a' else a
- If $b = +\infty$ then b' else b

➥ Refine open bounds

Iteration with narrowing

`1:  n = 0;`
`2:  while n < 1000 do`
`3:    n = n + 1;`
`4:  end`
`5:` ~~`exit;`~~ `t[n] = 0;`

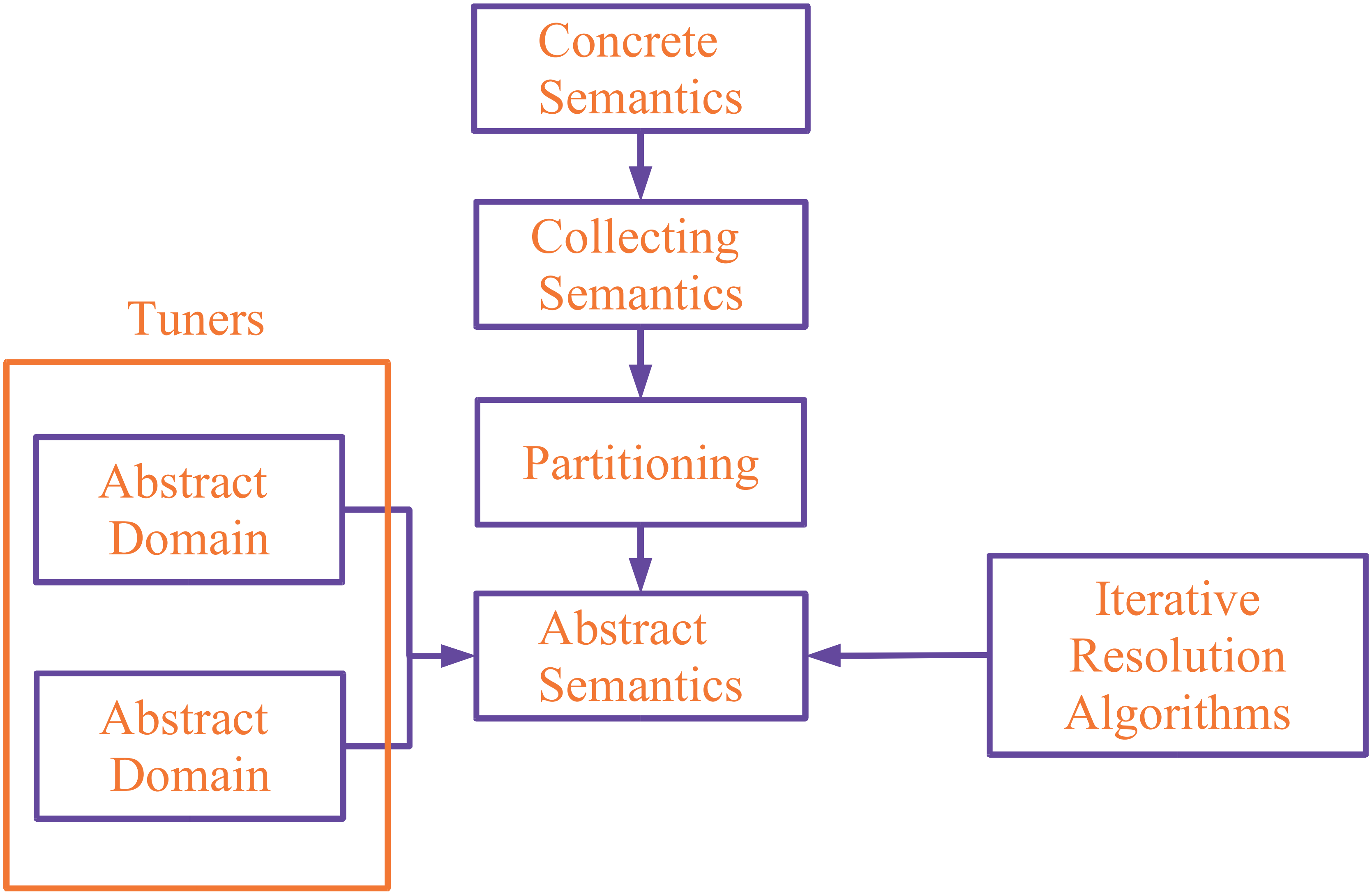
$$(E_2^{\#})_{n+1} = (E_2^{\#})_n \,\Delta\, \left([\texttt{n = 0}]^{\#}(E_1^{\#})_n \cup^{\#} (E_4^{\#})_n\right)$$

Beginning of iteration: $E_2^{\#} = [0, +\infty[$

Iteration 1: $E_2^{\#} = [0, 1000] \Rightarrow$ stable

Consequence: $E_5^{\#} = [1000, 1000]$

Methodology



Tuning the abstract domains

```
1:  n = 0;
2:  k = 0;
3:  while n < 1000 do
4:    n = n + 1;
5:    k = k + 1;
6:  end
7:  exit
```

- Intervals:

$$E_4^\# = \langle n \Rightarrow [0, 1000], k \Rightarrow [0, +\infty[\rangle$$

- Convex polyhedra or DBMs:

$$E_4^\# = \langle 0 \leq n \leq 1000, 0 \leq k \leq 1000, n - k = 0 \rangle$$