Indirect M-MRAC for Systems with Time Varying Parameters and Bounded Disturbances

Vahram Stepanyan and Kalmanje Krishnakumar

***Abstract*— The paper presents a prediction-identification model based adaptive control method for uncertain systems with time varying parameters in the presence of bounded external disturbances. The method guarantees desired tracking performance for the system's state and input signals. This is achieved by feeding back the state prediction error to the identification model. It is shown that the desired closed-loop properties are obtained with fast adaptation when the error feedback gain is selected proportional to the square root of the adaptation rate. The theoretical findings are confirmed via a simulation example.**

I. INTRODUCTION

Adaptive control has been considered as a promising technology to improve stability and performance of uncertain systems. However, limitations of conventional adaptive methods (see for example [1]) have prevented them to be widely adopted in safety-critical systems.

During past two decades majority of the efforts have been directed to improving the transient of the tracking error (see for example recent results in [2], [4], [5], [9] for the systems with time varying uncertainties), but not the control signal, the behavior of which significantly contributes to the aforementioned limitations.

These limitations have been addressed in the $\mathcal{L}_1$ adaptive control framework [3]. It has been shown that the desired transient can be obtained via fast adaptation and a low-pass filter, which a priori sets the bandwidth, within which the uncertainties in the system can be compensated for.

An alternative method, which guarantees desired transient behavior of the closed-loop system, has been proposed in [8]. It is based on the modification of the reference model by the tracking error feedback, and is called modified reference model MRAC (M-MRAC). The idea behind the method was to drive the reference model toward the system proportional to the tracking error, thus preventing the system's attempt to aggressively maneuver toward the reference model in the initial stage of the process. It turns out that the error feedback gain determines the damping in the control signal dynamics, whereas the adaptation rate determines the frequency. This allows the designer to choose proper values for the parameters.

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In this paper, we extend the approach to the systems with time varying parameters using a prediction (or identification) model based approach. The parameters are assumed to be bounded and vary with bounded derivatives, no matter how large these bounds are. The parameter estimates are generated using the state prediction error as in the case of conventional indirect adaptive control schemes, which is the reason to name the approach indirect M-MRAC. However, our prediction model differs from the conventional ones by a prediction error feedback term, which turns out to play the same role as the tracking error feedback term plays in the direct M-MRAC approach. Hence, the desired closed-loop behavior can be achieved with fast adaptation by selecting a proper error feedback gain, which also separates the time scale of the adaptive estimation from that of the system's dynamics.

The rest of the paper outlines the properties of the proposed indirect M-MRAC control architecture and demonstrates the benefits of it in a simulation example.

II. PROBLEM STATEMENT

Consider the system

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\Lambda(t)\left[\boldsymbol{u}(t) + K(t)\boldsymbol{g}(\boldsymbol{x}(t)) + \boldsymbol{d}(t)\right] \quad (1)$$

with $\boldsymbol{x}(0) = \boldsymbol{x}_0$, where $\boldsymbol{x} \in R^n$ is the state of the system, $\boldsymbol{u} \in R^q$ is the control, $A \in R^{n\times n}$ and $B \in R^{n\times q}$ are given constant matrices with A being Hurwitz and (A, B) controllable, $\boldsymbol{g} : R^n \to R^p$ is continuously differentiable, $\Lambda : R^+ \to R^{q\times q}$ is positive definite with bounded and piecewise continuous unknown entries, which have bounded derivatives, $K : R^+ \to R^{q\times p}$ is an unknown parameter matrix with bounded and piecewise continuous entries, which have bounded derivatives, and $\boldsymbol{d} : R^+ \to R^q$ is bounded and piecewise continuous disturbance with a bounded derivative.

The control objective is to design a control input such that the system (1) tracks the reference model.

$$\dot{\boldsymbol{x}}_m(t) = A\boldsymbol{x}_m(t) + B\boldsymbol{r}(t) \quad (2)$$

with $\boldsymbol{x}_m(0) = \boldsymbol{x}_{m0}$, where $\boldsymbol{r} : R^+ \to R^q$ is bounded and piecewise continuous command with a bounded derivative.

We notice that the system (1) can be represented in the form

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{r}(t)\\ &+ B\Lambda(t)\left[\boldsymbol{u}(t) + \Theta(t)\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{r}) + \boldsymbol{d}(t)\right]\,, \end{aligned} \quad (3)$$

where $\Theta(t)\boldsymbol{f}(\boldsymbol{x},\boldsymbol{r}) = K(t)\boldsymbol{g}(\boldsymbol{x}(t)) - \Lambda^{-1}(t)\boldsymbol{r}(t)$. Let

$$\begin{aligned}&\|\Lambda(t)\|_{\mathcal{L}_\infty} \le \lambda^*,\ \|\Theta(t)\|_{\mathcal{L}_\infty} \le \vartheta^*,\ \|\boldsymbol{d}(t)\|_{\mathcal{L}_\infty} \le d^* \\ &\|\dot{\Lambda}(t)\|_{\mathcal{L}_\infty} \le \lambda_d^*,\ \|\dot{\Theta}(t)\|_{\mathcal{L}_\infty} \le \vartheta_d^*,\ \|\dot{\boldsymbol{d}}(t)\|_{\mathcal{L}_\infty} \le d_d^* \end{aligned} \tag{4}$$

III. Prediction Model

We introduce the following adaptive prediction model

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}(t) &= A\hat{\boldsymbol{x}}(t) + B\boldsymbol{r}(t) \\ &+ B\hat{\Lambda}(t)[\boldsymbol{u}(t) + \hat{\Theta}(t)\boldsymbol{f}(\boldsymbol{x},\boldsymbol{r}) + \hat{\boldsymbol{d}}(t)] + k\tilde{\boldsymbol{x}}(t)\end{aligned} \tag{5}$$

with $\hat{\boldsymbol{x}}(0) = \hat{\boldsymbol{x}}_0$, where $\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$ is the prediction error, $k > 0$ is a design parameter, $\hat{\Lambda}(t)$, $\hat{\Theta}(t)$ and $\hat{\boldsymbol{d}}(t)$ are the estimates of the unknown quantities, generated according to adaptive laws

$$\begin{aligned}\dot{\hat{\Theta}}(t) &= \gamma \operatorname{Pr}\left(\hat{\Theta}(t),\ B^\top P\tilde{\boldsymbol{x}}(t)\boldsymbol{f}^\top(\boldsymbol{x},\boldsymbol{r})\right) \\ \dot{\hat{\Lambda}}(t) &= \gamma \operatorname{Pr}\left(\hat{\Lambda}(t),\ B^\top P\tilde{\boldsymbol{x}}(t)[\boldsymbol{u}(t) + \hat{\Theta}(t)\boldsymbol{f}(\boldsymbol{x},\boldsymbol{r})]^\top\right) \\ \dot{\hat{\boldsymbol{d}}}(t) &= \gamma \operatorname{Pr}\left(\hat{\boldsymbol{d}}(t),\ B^\top P\tilde{\boldsymbol{x}}(t)\right),\end{aligned} \tag{6}$$

where $\gamma > 0$ is the adaptation rate, $P = P^\top > 0$ is the solution of the Lyapunov equation $A^\top P + PA = -Q$ for some $Q = Q^\top > 0$, and $\operatorname{Pr}(\cdot,\cdot)$ denotes the projection operator [7], which is defined as $\operatorname{Pr}(\hat{\theta}, \boldsymbol{y}) = [\mathbb{I} - G(\hat{\theta})]\boldsymbol{y}$, where

$$G(\hat{\theta}) = \begin{cases} 0, & \text{if } \varphi(\hat{\theta}) < 0 \\ 0, & \text{if } \varphi(\hat{\theta}) \ge 0,\ \nabla\varphi^\top(\hat{\theta})\boldsymbol{y} \le 0 \\ \frac{\nabla\varphi(\hat{\theta})\nabla\varphi^\top(\hat{\theta})}{\|\nabla\varphi(\hat{\theta})\|^2}\varphi(\hat{\theta}), & \text{if } \varphi(\hat{\theta}) \ge 0,\ \nabla\varphi^\top(\hat{\theta})\boldsymbol{y} > 0 \end{cases}$$

with the notation $\nabla\varphi(\hat{\theta}) = \frac{\partial\varphi(\hat{\theta})}{\partial\hat{\theta}}$, and the smooth convex functions $\varphi(\hat{\theta})$ is given by $\varphi(\hat{\theta}) = \frac{\operatorname{tr}(\hat{\theta}^\top\hat{\theta}) - \theta_{\max}^2}{\epsilon_\theta\theta_{\max}^2}$ with $\theta_{\max}$ denoting the norm bound imposed on the parameter matrix $\hat{\theta}$ and ϵ_θ denoting the convergence tolerance. The projection operator has the following properties

Lemma 3.1: [7] Let $\theta_0 \in \Omega_0 = \{\hat{\theta} \in \mathbb{R}^n \mid \varphi(\hat{\theta}) \le 0\}$, and let the parameter $\hat{\theta}(t)$ evolve according to the dynamics

$$\dot{\hat{\theta}}(t) = \operatorname{Pr}(\hat{\theta}(t),\ \boldsymbol{y}), \quad \hat{\theta}(t_0) \in \Omega\,. \tag{7}$$

Then 1) $\hat{\theta}(t) \in \Omega_1 = \{\hat{\theta} \in \mathbb{R}^n \mid \varphi(\hat{\theta}) \le 1\}$ or $\|\hat{\theta}(t)\| \le \theta^*$ for all $t \ge t_0$, where $\theta^* = \sqrt{1+\epsilon_\theta}\ \theta_{\max}$, 2) $[\hat{\theta}(t) - \theta_0]^\top[\operatorname{Pr}(\hat{\theta}(t),\ \boldsymbol{y}) - \boldsymbol{y}] \le 0$ for all $t \ge t_0$.

It is straightforward to verify that

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}(t) &= (A - k\mathbb{I})\tilde{\boldsymbol{x}}(t) + B\Lambda(t)[\tilde{\Theta}(t)\boldsymbol{f}(\boldsymbol{x},\boldsymbol{r}) + \tilde{\boldsymbol{d}}(t)] \\ &+ B\tilde{\Lambda}(t)[\boldsymbol{u}(t) + \hat{\Theta}(t)\boldsymbol{f}(\boldsymbol{x},\boldsymbol{r}) + \hat{\boldsymbol{d}}(t)]\,,\end{aligned} \tag{8}$$

where $\tilde{\Theta}(t) = \Theta(t) - \hat{\Theta}(t)$, $\tilde{\Lambda}(t) = \Lambda(t) - \hat{\Lambda}(t)$ and $\tilde{\boldsymbol{d}}(t) = \boldsymbol{d}(t) - \hat{\boldsymbol{d}}(t)$ are the estimation errors.

Lemma 3.2: If $\hat{\boldsymbol{x}}_0 = \boldsymbol{x}_0$, then the prediction error $\tilde{\boldsymbol{x}}(t)$ satisfies the bound

$$\|\tilde{\boldsymbol{x}}(t)\| \le \sqrt{\frac{c}{\lambda_{\min}(P)}}\frac{1}{\sqrt{\gamma}}\,, \tag{9}$$

where $c = c_1 + \frac{c_2}{2k}$, $c_1 = 4\lambda^* d^{*2} + 4\lambda^*\vartheta^{*2} + 4\lambda^{*2}$, and $c_2 = 4\lambda^*\vartheta^*\vartheta_d^* + 4\lambda^* d^* d_d^* + 4\lambda_d^* d^{*2} + 4\lambda_d^*\vartheta^{*2}$.

Proof: The derivative of the candidate Lyapunov function

$$\begin{aligned}V(t) &= \tilde{\boldsymbol{x}}^\top(t)P\tilde{\boldsymbol{x}}(t) + \gamma^{-1}\tilde{\boldsymbol{d}}^\top(t)\Lambda(t)\tilde{\boldsymbol{d}}(t) \\ &+ \gamma^{-1}\operatorname{tr}\left(\tilde{\Theta}^\top(t)\Lambda(t)\tilde{\Theta}(t) + \tilde{\Lambda}^\top(t)\tilde{\Lambda}(t)\right),\end{aligned} \tag{10}$$

computed along the trajectories of the prediction error dynamics (8) and the adaptive laws (6), satisfies the inequality

$$\begin{aligned}\dot{V}(t) &\le -\tilde{\boldsymbol{x}}^\top(t)Q\tilde{\boldsymbol{x}}(t) - 2k\tilde{\boldsymbol{x}}^\top(t)P\tilde{\boldsymbol{x}}(t) \\ &+ 2\gamma^{-1}\operatorname{tr}\big(\dot{\Theta}^\top(t)\Lambda(t)\tilde{\Theta}(t)\big) + 2\gamma^{-1}\dot{\boldsymbol{d}}^\top(t)\Lambda(t)\tilde{\boldsymbol{d}}(t) \\ &+ \gamma^{-1}\tilde{\boldsymbol{d}}^\top(t)\dot{\Lambda}(t)\tilde{\boldsymbol{d}}(t) + \gamma^{-1}\operatorname{tr}\left(\tilde{\Theta}^\top(t)\dot{\Lambda}(t)\tilde{\Theta}(t)\right).\end{aligned} \tag{11}$$

The projection operator in the adaptive laws (6) guarantees the inequalities $\|\hat{\Theta}(t)\| \le \vartheta^*$, $\|\hat{\Lambda}(t)\| \le \lambda^*$ and $\|\hat{\boldsymbol{d}}(t)\| \le d^*$. Therefore $\|\tilde{\Theta}(t)\| \le 2\vartheta^*$, $\|\tilde{\Lambda}(t)\| \le 2\lambda^*$, $\|\tilde{\boldsymbol{d}}(t)\| \le 2d^*$ and $2\operatorname{tr}\big(\dot{\Theta}^\top(t)\Lambda(t)\tilde{\Theta}(t)\big) + 2\dot{\boldsymbol{d}}^\top(t)\Lambda(t)\tilde{\boldsymbol{d}}(t) + \tilde{\boldsymbol{d}}^\top(t)\dot{\Lambda}(t)\tilde{\boldsymbol{d}}(t) + \operatorname{tr}\left(\tilde{\Theta}^\top(t)\dot{\Lambda}(t)\tilde{\Theta}(t)\right) \le c_2$. That is

$$\dot{V}(t) \le -\tilde{\boldsymbol{x}}^\top(t)Q\tilde{\boldsymbol{x}}(t) - 2k\tilde{\boldsymbol{x}}^\top(t)P\tilde{\boldsymbol{x}}(t) + \gamma^{-1}c_2\,. \tag{12}$$

On the other hand we have $\tilde{\boldsymbol{d}}^\top(t)\Lambda(t)\tilde{\boldsymbol{d}}(t) + \operatorname{tr}\left(\tilde{\Theta}^\top(t)\Lambda(t)\tilde{\Theta}(t) + \tilde{\Lambda}^\top(t)\tilde{\Lambda}(t)\right) \le c_1$. It follows that $V(t) \le \tilde{\boldsymbol{x}}^\top(t)P\tilde{\boldsymbol{x}}^\top(t) + \gamma^{-1}c_1$. Therefore, if $V(\tau) > \gamma^{-1}c$, for some τ then $\tilde{\boldsymbol{x}}^\top(\tau)P\tilde{\boldsymbol{x}}^\top(\tau) > (2k\gamma)^{-1}c_2$, which implies that $\dot{V}(\tau) < 0$. Since $\tilde{\boldsymbol{x}}(0) = 0$ it follows that $V(0) \le \gamma^{-1}c_1 < \gamma^{-1}c$. Therefore $V(t) \le \gamma^{-1}c$ for all $t \ge 0$.

Since $\|\tilde{\boldsymbol{x}}(t)\|^2 \le \tilde{\boldsymbol{x}}^\top(t)P\tilde{\boldsymbol{x}}(t)/\lambda_{\min}(P) \le V(t)/\lambda_{\min}(P)$, the inequality (9) follows. ■

It can be observed from Lemma 3.2 that the state prediction error can be decreased as desired by increasing the adaptation rate γ, when the prediction model is precisely initialized. The next lemma shows that the initialization error results in an additive exponentially decaying term.

Lemma 3.3: If $\hat{\boldsymbol{x}}_0 \ne \boldsymbol{x}_0$, then $\tilde{\boldsymbol{x}}(t)$ satisfies the bound

$$\|\tilde{\boldsymbol{x}}(t)\| \le \sqrt{\frac{c_3}{\lambda_{\min}(P)}}e^{-kt} + \sqrt{\frac{c}{\gamma\lambda_{\min}(P)}}\,, \tag{13}$$

where $c_3 = |V(0) - \frac{c}{\gamma}|$, and $V(t)$ is defined by (10).

Proof: Using the same $V(t)$ as in Lemma 3.2 and following the same steps one can arrive to the inequality

$$\dot{V}(t) \le -2k[V(t) - \gamma^{-1}c_1] + \gamma^{-1}c_2\,, \tag{14}$$

integration of which results in

$$V(t) \le \left[V(0) - \frac{c}{\gamma}\right]e^{-2kt} + \frac{c}{\gamma} \le c_3e^{-2kt} + \frac{c}{\gamma}\,. \tag{15}$$

Recalling that $\|\tilde{\boldsymbol{x}}(t)\|^2 \le V(t)/\lambda_{\min}(P)$, we readily obtain

$$\|\tilde{\boldsymbol{x}}(t)\| \le \sqrt{\frac{1}{\lambda_{\min}(P)}}\sqrt{c_3e^{-kt} + \frac{c}{\gamma}}\,, \tag{16}$$

Taking into account the inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for any $a \ge 0,\ b \ge 0$, the bound (13) is concluded. ■

Since the effect of the prediction model initialization error decays exponentially with the rate k, which is assumed to be set to large values for the fast adaptation, in the next derivations we assume that $\hat{\boldsymbol{x}}_0 = \boldsymbol{x}_0$.

IV. Control Design

Since the reference model is designed to satisfy the robustness and performance specifications, one would naturally select the control signal

$$\boldsymbol{u}(t) = -\Theta(t)\boldsymbol{f}(\boldsymbol{x},\boldsymbol{r}) - \boldsymbol{d}(t)\,. \tag{17}$$

to achieve the control objective, if the system (3) were completely known. Obviously, $\boldsymbol{u}(t)$ is not implementable, therefore its adaptive version

$$\hat{\boldsymbol{u}}(t) = -\hat{\Theta}^{\top}(t)\boldsymbol{f}(\boldsymbol{x},\boldsymbol{r}) - \hat{\boldsymbol{d}}(t) \tag{18}$$

is used. When this control signal is applied, the prediction model (5) reduces to the modified reference model introduced in the M-MRAC architecture, that is

$$\dot{\hat{\boldsymbol{x}}}(t) = A\hat{\boldsymbol{x}}(t) + B\boldsymbol{r}(t) + k\tilde{\boldsymbol{x}}(t)\,, \tag{19}$$

Lemma 4.1: Let the system (3) be controlled by the adaptive control $\hat{\boldsymbol{u}}(t)$ (18). Then closed loop signals are bounded.

Proof: Under the action of the adaptive control $\hat{\boldsymbol{u}}(t)$, the error between the prediction model and reference model $\boldsymbol{e}_m(t) = \hat{\boldsymbol{x}}(t) - \boldsymbol{x}_m(t)$ satisfies the equation

$$\dot{\boldsymbol{e}}_m(t) = A\boldsymbol{e}_m(t) + \lambda\tilde{\boldsymbol{x}}(t)\,, \tag{20}$$

Since A is Hurwitz, and $\tilde{\boldsymbol{x}}(t)$ is bounded according to Lemma 3.2, it follows that $\boldsymbol{e}_m(t)$ is bounded. Since the input $\boldsymbol{r}(t)$ is bounded, the reference model's state $\boldsymbol{x}_m(t)$ is bounded, therefore the predicted state $\hat{\boldsymbol{x}}(t)$ is bounded as well. Then, it follows that the system's state $\boldsymbol{x}(t)$ is bounded. The parameter estimates are guaranteed to be bounded by the projection operator, therefore $\hat{\boldsymbol{u}}(t)$ is also bounded. ■

Lemma 4.2: Let the system (3) be controlled by the controller (18), which is defined by the prediction model (5) and the adaptive law (6). Then

$$\|\tilde{\boldsymbol{u}}(t)\| \le \beta_1 e^{-\nu_1 t} + \beta_2\gamma^{-\frac{1}{2}}\,, \tag{21}$$

where $\tilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t) - \hat{\boldsymbol{u}}(t)$, and ν_1, β_1 and β_2 are positive constants to be specified in the proof.

Proof: It is easy to show that $\tilde{\boldsymbol{u}}(t)$ satisfies the equation

$$\begin{bmatrix} \dot{\tilde{\boldsymbol{u}}}(t) \\ \ddot{\tilde{\boldsymbol{u}}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{I}_{q\times q} \\ -\gamma F(t)L(t) & -k\mathbb{I}_{q\times q} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{u}}(t) \\ \dot{\tilde{\boldsymbol{u}}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \boldsymbol{z}_1(t) + \begin{bmatrix} 1 \\ k \end{bmatrix} \boldsymbol{z}_2(t)\,, \tag{22}$$

where we denote $\rho(t) = \boldsymbol{f}^{\top}(\boldsymbol{x},\boldsymbol{r})\boldsymbol{f}(\boldsymbol{x},\boldsymbol{r}) + 1$, $F(t) = \rho(t)\mathbb{I}_{q\times q} - H(t)$, $H(t) = G(\hat{\Theta})\boldsymbol{f}^{\top}(\boldsymbol{x},\boldsymbol{r})\boldsymbol{f}(\boldsymbol{x},\boldsymbol{r}) + G(\hat{\boldsymbol{d}})$, $L = B^{\top}PB\Lambda(t)$ ($L(t)$ is positive definite), $\boldsymbol{z}_1(t) = [\dot{\rho}(t)B_0^{\top}P + \rho(t)B_0^{\top}PA_m]\tilde{\boldsymbol{x}}(t)$, and $\boldsymbol{z}_2(t) = -\tilde{\Theta}^{\top}(t)\dot{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{r}) - \dot{\boldsymbol{d}}(t)$. Since $\boldsymbol{x}(t)$ is bounded, it follows that $\rho(t)$ is bounded. That is, there exists a positive constant α_1 such that $1 \le \rho(t) \le \alpha_1$ for all $0 \le t < \infty$. On the other hand, it follows from the definition of the projection operator that $\|G(\hat{\Theta})\| \le 1$ and $\|G(\hat{\boldsymbol{d}})\| \le 1$. Therefore $F(t)$ is bounded. Further, it follows from the dynamics (3) that $\dot{\boldsymbol{x}}(t)$ is bounded. Therefore $\dot{\rho}(t)$ and $\boldsymbol{z}_2(t)$ are bounded. That is, there exist positive constants α_2 and α_3 such that $|\dot{\rho}(t)| \le \alpha_2$ and $\|\boldsymbol{z}_2(t)\| \le \alpha_3$ for all $0 \le t < \infty$. Hence, (22) can be considered as a second order linear system in $\tilde{\boldsymbol{u}}(t)$ with time varying coefficients, where the adaptation rate γ determines the frequency of $\tilde{\boldsymbol{u}}(t)$ and the feedback gain k determines the damping ratio. We notice that selection of the initial parameter estimates inside the convex sets defined by the projection operator results in $H(t) = 0$ on some initial interval $[0\ t_1]$. Therefore, $F(t) = \rho(t)\mathbb{I}_q$ on $[0\ t_1]$. Let $a_0 = \frac{\alpha_1\lambda^0 + \lambda_0}{2}$, where $\lambda^0 = \max_{t\ge 0}\lambda(L(t))$ and $\lambda_0 = \min_{t\ge 0}\lambda(L(t))$. Denoting $E(t) = a_0\mathbb{I}_q - \rho(t)L(t)$, we can write

$$\begin{bmatrix} \dot{\tilde{\boldsymbol{u}}}(t) \\ \ddot{\tilde{\boldsymbol{u}}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{I}_{q\times q} \\ -\gamma a_0\mathbb{I}_{q\times q} & -k\mathbb{I}_{q\times q} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{u}}(t) \\ \dot{\tilde{\boldsymbol{u}}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \boldsymbol{z}_1(t) + \begin{bmatrix} 1 \\ k \end{bmatrix} \boldsymbol{z}_2(t) + \gamma \begin{bmatrix} 0_{n\times n} \\ E(t) \end{bmatrix} \tilde{\boldsymbol{u}}(t)\,, \tag{23}$$

the solution of which has an equivalent integral form

$$\tilde{\boldsymbol{u}}(t) = \boldsymbol{\psi}(t)\left[\tilde{\boldsymbol{u}}(0)\quad \dot{\tilde{\boldsymbol{u}}}(0)\right]^{\top} + \gamma\int_0^t \psi_2(t-\tau)[\boldsymbol{z}_1(\tau) + E(\tau)\tilde{\boldsymbol{u}}(\tau)]d\tau + \int_0^t[\psi_1(t-\tau) + k\psi_2(t-\tau)]\boldsymbol{z}_2(\tau)d\tau \tag{24}$$

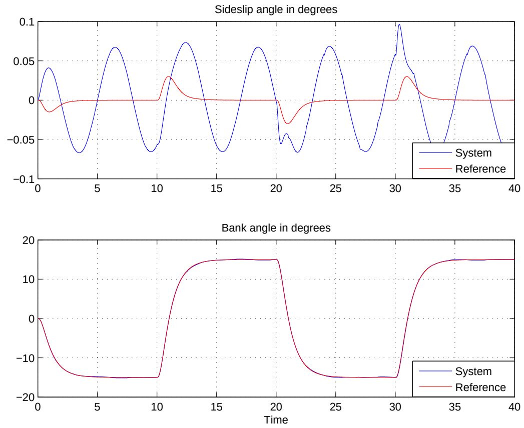

Fig. 1. M-MRAC angle tracking performance with $\gamma = 1000$.

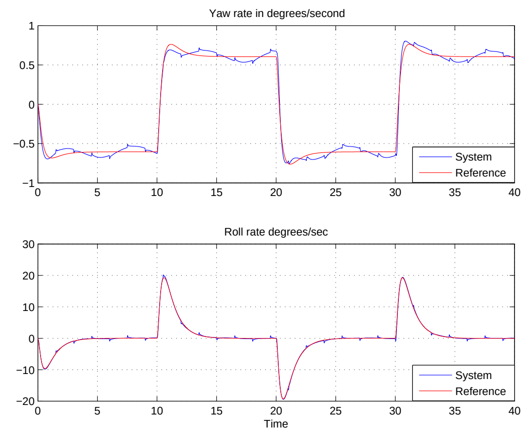

Fig. 2. M-MRAC rate tracking performance with $\gamma = 1000$.

where $\boldsymbol{\psi}(t) = [\psi_1(t) \ \ \psi_2(t)]$ is the first row of the transition matrix of the LTI part of system (23). Following [8], we select

$$k = 2\sqrt{\gamma a_0}\,, \tag{25}$$

which results in the minimum norm $\|\psi_2(t)\|_{\mathcal{L}_1} = (\gamma a_0)^{-1}$. For the same k, we have $\|\psi_1(t) + k\psi_2(t)\|_{\mathcal{L}_1} \leq 4(\gamma a_0)^{-1/2}$. Since $\|E(t)\|_{\mathcal{L}_\infty} = a_0 - \lambda_0$, we obtain $1 - \gamma\|\psi_2(t)\|_{\mathcal{L}_1}\|E(t)\|_{\mathcal{L}_\infty} = 2\lambda_0 a_0^{-1}$. Then, according to [8], it follows from the expression (24) that

$$\begin{aligned} \|\tilde{\boldsymbol{u}}(t)\| &\leq b_1(\|\tilde{\boldsymbol{u}}(0)\| + \|\dot{\tilde{\boldsymbol{u}}}(0)\|)e^{-\nu_1 t} \\ &+ \frac{1}{2\lambda_0}\|\boldsymbol{z}_1(t)\|_{\mathcal{L}_\infty} + \frac{4\sqrt{a_0}}{2\lambda_0}\|\boldsymbol{z}_2(t)\|_{\mathcal{L}_\infty}\,, \end{aligned} \tag{26}$$

where b_1 is a positive constant and

$$\nu_1 = -\frac{\sqrt{\gamma}}{2}\left(\sqrt{a_0} - \sqrt{a_0 - \lambda_0}\right). \tag{27}$$

From the definition of $\boldsymbol{z}_1(t)$ and Lemma 3.2 we have

$$\|\boldsymbol{z}_1(t)\|_{\mathcal{L}_\infty} \leq (\alpha_2\|B^\top P\| + \alpha_1\|B^\top PA\|)\sqrt{\frac{c}{\gamma\lambda_{\min}(P)}}\,.$$

Then, it is straightforward to obtain the bound (21) with

$$\begin{aligned} \beta_1 &= \frac{b_1(\|\tilde{\boldsymbol{u}}(0)\| + \|\dot{\tilde{\boldsymbol{u}}}(0)\|)}{2\lambda_0} \\ \beta_2 &= \frac{4\alpha_3\sqrt{a_0}}{2\lambda_0} + \frac{(\alpha_2\|B^\top P\| + \alpha_1\|B^\top PA\|)\sqrt{c}}{2\lambda_0\sqrt{\lambda_{\min}(P)}}\,. \end{aligned}$$

This concludes the proof. ■

V. Tracking Error

In this section we derive a norm bound for the tracking error $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_m(t)$, which is given by the following lemma.

Lemma 5.1: Let the system (1) be controlled by the controller (18), which is defined by the prediction model (5) and the adaptive law (6). Then

$$\|\boldsymbol{e}(t)\| \leq \beta_3 e^{-\nu t} + \beta_4\gamma^{-\frac{1}{2}}\,, \tag{28}$$

β_3 and β_4 are positive constants to be specified in the proof.

Proof: It is straightforward to obtain the tracking error dynamics in the form

$$\dot{\boldsymbol{e}}(t) = A\boldsymbol{e}(t) - B\Lambda(t)\tilde{\boldsymbol{u}}(t)\,. \tag{29}$$

Since A is Hurwitz, it follows that there exist positive constants b_2 and ν_2 such that $\|e^{At}\| \leq b_2 e^{-\nu_2 t}$. Therefore the following bound can be obtained

$$\begin{aligned} &\|\boldsymbol{e}(t)\| \leq b_2\|\boldsymbol{e}(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|_{\mathcal{L}_\infty}\textstyle\int_0^t e^{-\nu_2(t-\tau)}\,\cdot \\ &[\beta_1 e^{-\nu_1\tau} + \beta_2\gamma^{-\frac{1}{2}}]d\tau \leq b_2\|\boldsymbol{e}(0)\|e^{-\nu_2 t} + b_2\|B\Lambda(t)\|_{\mathcal{L}_\infty} \\ &\left[\tfrac{\beta_1}{\nu_1-\nu_2}(e^{-\nu_2 t} - e^{-\nu_1 t}) + \tfrac{\beta_2}{\nu_2}(1 - e^{-\nu_2 t})\gamma^{-\frac{1}{2}}\right] \end{aligned} \tag{30}$$

which can be expressed in the form of (28) with

$$\begin{aligned} \beta_3 &= b_2\|\boldsymbol{e}(0)\| + \frac{b_2\beta_1}{|\nu_2 - \nu_1|}\|B\Lambda(t)\|_{\mathcal{L}_\infty} \\ \beta_4 &= \frac{1}{\nu_1}b_2\beta_2\|B\Lambda(t)\|_{\mathcal{L}_\infty}\,. \end{aligned}$$

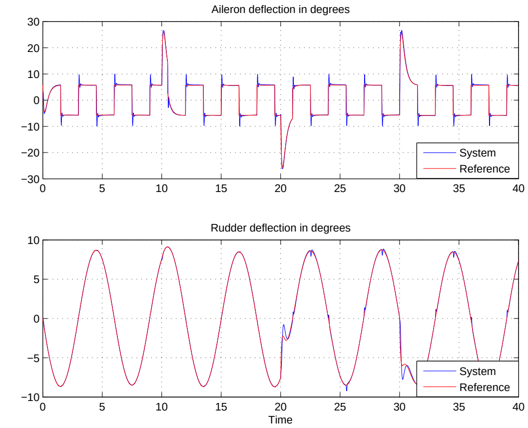

Fig. 3. M-MRAC control signal time history with $\gamma = 1000$.

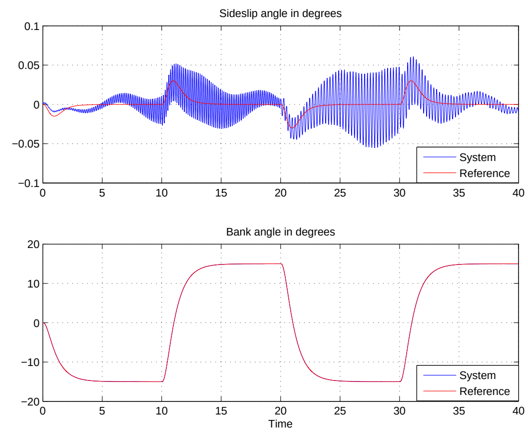

Fig. 4. MRAC angle tracking performance with $\gamma = 1000$.

and $\nu = \min[\nu_1, \nu_2]$. ■

Remark 5.1: The proposed method guarantees the regulation of all error signals by increasing the adaptation rate, which is only subject to available computational power. Therefore, with fast adaptation the control objective is achieved without generating unwanted excursions and oscillations in adaptive signals. The effects of the external disturbances and parameter variations are compensated for by the fast adaptation, and the effects of the initial conditions decay exponentially. □

Remark 5.2: It can be observed that the dynamics of the reference model, the operating system and the tracking error have the same time scales determined by the matrix A. Hence, the reference model initialization error generates an additive exponential term $b_2\|\boldsymbol{e}(0)\|e^{-\nu t}$ in the bound of the tracking error with a rate of decay defined by the time constant of the reference model, since the adaptation process is much faster. The time scale of the prediction error dynamics (8) is determined by k, which is proportional to $\sqrt{\gamma}$, and the time scale of the adaptive estimates is determined by γ. Therefore, for large values of γ the time scale of the adaptive estimation process is separated from the time scale of the underlying closed-loop dynamics, which is not achievable by conventional adaptive methods [1]. □

VI. Illustrative Example

In this section, the advantages of the proposed indirect M-MRAC architecture are demonstrated in simulations for a dynamic model that represents the lateral-directional motion of a generic transport aircraft (GTM) [6]. The nominal model is the linearized lateral-directional dynamics of GTM at the altitude of 30000 ft and speed of $0.8M$ and is given by

$$\dot{\boldsymbol{x}}(t) = A_n \boldsymbol{x}(t) + B_n \boldsymbol{u}(t)\,, \tag{31}$$

where $\boldsymbol{x} = [\beta \;\; r \;\; p \;\; \phi]^\top$ is the lateral-directional state vector, in which β is the sideslip angle, r is the yaw rate, p is the roll rate, ϕ is the bank angle, and $\boldsymbol{u} = [\delta_a \;\; \delta_r]^\top$ is the control signal that includes the aileron deflection δ_a and the rudder deflection δ_r, and the numerical values for A_n and B_n are

$$A_n = \begin{bmatrix} -0.1578 & -0.9907 & 0.0475 & 0.0404 \\ 2.7698 & -0.3842 & 0.0240 & 0 \\ -10.1076 & 0.5090 & -1.7520 & 0 \\ 0 & 0.0506 & 1.0000 & 0 \end{bmatrix},$$

$$B_n = \begin{bmatrix} 0.0042 & 0.0476 \\ 0.0351 & -2.2464 \\ 6.3300 & 1.7350 \\ 0 & 0 \end{bmatrix}.$$

The reference model is selected from the perspective of improving the performance characteristics of the nominal dynamics and is given by the equation (2), where $A = A_n - B_n K$ and $B = B_n N$, with the feedback and feedforward matrices

$$K = \begin{bmatrix} 0 & 0 & 0.43 & 0.55 \\ 1.92 & -1.5 & 0 & 0 \end{bmatrix}, N = \begin{bmatrix} 1.26 & 0.65 \\ 3.33 & -0.07 \end{bmatrix}.$$

The reference model is driven by a command, which is chosen to be a series of coordinated turn maneuvers. That is, sideslip angle command is set to zero and the bank angle command is chosen to be a square wave of the amplitude of 15 degrees and of the frequency $\frac{\pi}{10}$ rad/sec, which is filtered through a first order stable filter $\frac{10}{s+10}$.

The uncertain model of GTM roughly corresponds to 28% loss of left wing tip at $t = 0$ sec, and 55% loss of rudder surface and vertical tail at $t = 20$ sec. Its dynamics are in the form of the equation (3) with piecewise constant $\Theta(t)$ and $\Lambda(t)$, and $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$. The corresponding numerical values are

$$\Theta(t) = \begin{cases} \begin{bmatrix} -0.1820 & 0.0149 & -0.1049 & 0 \\ 0.0807 & -0.0109 & 0.0168 & 0 \end{bmatrix}, & t \le 20 \\[2ex] \begin{bmatrix} -0.2268 & 0.0209 & -0.1053 & 0 \\ -0.8514 & 0.0692 & 0.0003 & 0 \end{bmatrix}, & t > 20 \end{cases}$$

$$\Lambda(t) = \begin{cases} \begin{bmatrix} 0.5401 & 0.0167 \\ -0.0632 & 1.0524 \end{bmatrix}, & t \le 20 \\[2ex] \begin{bmatrix} 0.5413 & -0.0492 \\ 0.0408 & 0.4225 \end{bmatrix}, & t > 20 \end{cases}$$

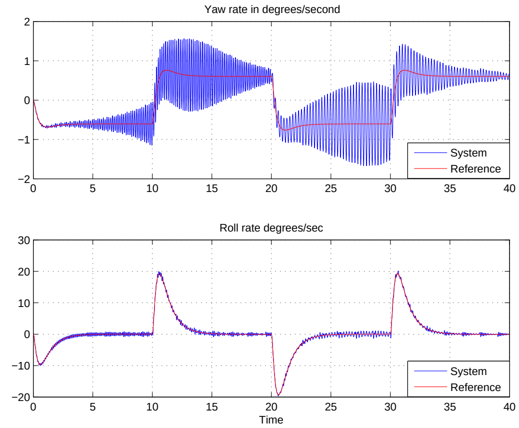

Fig. 5. MRAC rate tracking performance with $\gamma = 1000$.

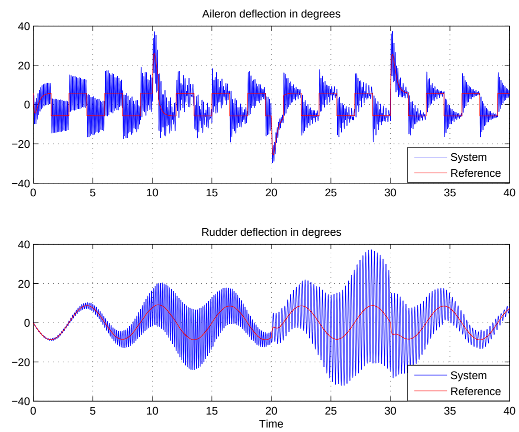

Fig. 6. MRAC control signal time history with $\gamma = 1000$.

The external disturbance is chosen to be a sinusoid of amplitude 0.1 and frequency $2\pi/3$ rad/sec in the yaw channel and a square wave of amplitude 0.15 and frequency $\pi/3$ rad/sec in the roll channel. The disturbance magnitude corresponds to 8.6 degrees of aileron deflection and 5.7 degrees of rudder deflection. In the definition of the projection operator the conservative bounds $\lambda^* = \vartheta^* = d^* = 10$ are used.

First, a simulation is run with $\gamma = 1000$, $Q = \mathbb{I}_4$ and k is computed according to (25), where we used conservative bounds $\lambda_0 = 0.2$ and $\lambda^0 = 2$. Figures 1 and 2 display the tracking performance of the states. Clearly good tracking is achieved with the chosen controller gains, for which the control time history is presented in Figure 3. It can be observed that the adaptive control signal closely follows the reference signal given by (17). Small spikes in the control signal are attributed to the discontinuities of the disturbance. For the comparison purposes we also present the conventional MRAC performance with the same setup. It can be observed from the Figure 4 that MRAC achieves output tracking with small oscillations in sideslip angle. However, the rates and the control surface deflection commands are experiencing unacceptable oscillations (see Figures 5 and 6).

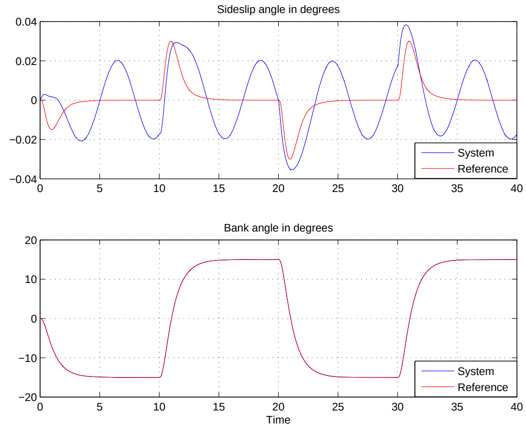

Fig. 7. M-MRAC output tracking with $\gamma = 10000$.

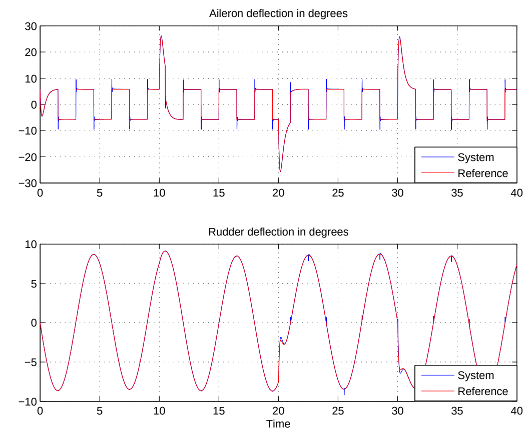

Fig. 8. M-MRAC input tracking with $\gamma = 10000$.

Next we increase the adaptation rate 10 fold. As it can be viewed form Figures 7 and 8, the output and input tracking performances are substantially improved as predicted. Computations show that the tracking error is decreased more than $\sqrt{10}$ fold, implying the the derived bounds are conservative. Farther increase of adaptation rate to $\gamma = 100000$ further improves the system's input and output performance (see Figures 9 and 10), which verifies the theoretical derivations.

VII. Conclusions

We have presented indirect modified reference model MRAC (M-MRAC) approach to uncertain systems with time varying parameters and bounded external disturbances without imposing "slow variation" restriction on the system's parameters. The method uses a prediction error feedback term to speed up the adaptive estimation process, which results in predictable transient behavior for both state and input variables of the system. It has been shown that the unwanted high frequency effects of the fast adaptation in the control signal can be regulated by the proper choice of the feedback gain.

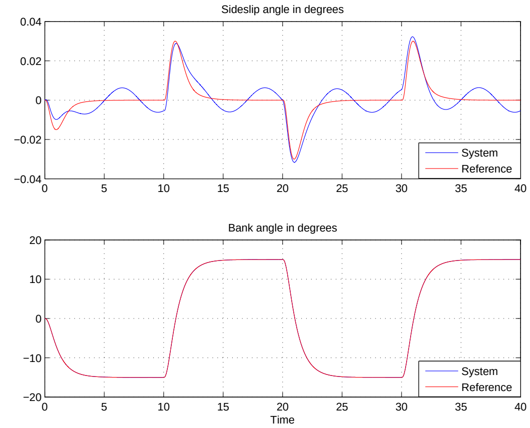

Fig. 9. M-MRAC output tracking with $\gamma = 100000$.

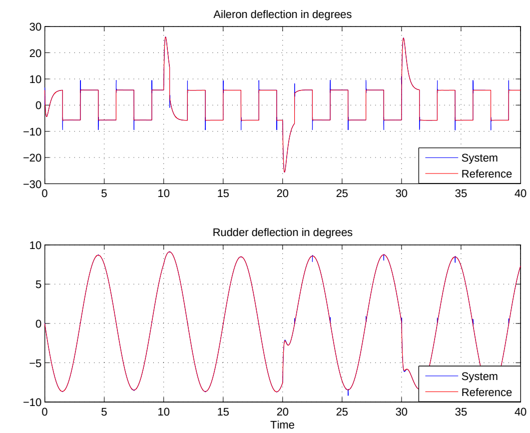

Fig. 10. M-MRAC input tracking $\gamma = 100000$.

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